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On the motion from rest of a class of cylinders. Elliptic cylinder case

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Using a very convenient calculation technique, Watson [1] investigated the unsteady boundary layer on a cylinder started from rest in a viscous incompressible fluid with the prescribed velocity

$$\Omega(t)=At^{\alpha}, \tag{1}$$

where t is the time measured from the starting, A and $\alpha>-1/2$ are constants In this manner, he gave a uniform treatment of both Blasius' cases [2]: that of the impulsive starting ($\alpha=0$) and that of the uniformly accelerated movement ($\alpha=1$). The Gortler's results [3] for integral values of α follow from his theory also. All these considerations based on the Prandtl's equations are closed in the frame of a thin boundary layer approximation to the Navier-Stokes equations. For finite Reynolds numbers they are in error due to the neglect of the wall curvature and the interaction between the boundary layer and the outer inviscid flow. This fact is emphasized by Wang [4, 5], whose examinations are restricted on impulsively started bodies. In this paper his results are extended on the motion from rest with a velocity following the power law (1).

Formulation

If one normalizes the Navier-Stokes and continuity equations by a characteristic length L_0, a characteristic time T_0 and a characteristic velocity U_0, they become

$$\frac{\partial \mathbf{v}}{\partial t}+\frac{1}{S}(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla p-\frac{1}{RS}\nabla\times(\nabla\times\mathbf{v}), \tag{2}$$

$$\nabla\cdot\mathbf{v}=0. \tag{3}$$

There $R=U_0L_0/\nu$ is the Reynolds number, $S=L_0/U_0T_0$ is the Strouhal number, $\mathbf{v}$, t and ν are the velocity, the time and the viscosity, respectively. The pressure p is normalized by $\varrho' U_0^2 S$, where ϱ' is the density. The last takes a place if one assumes that for the times into consideration the pressure forces are of the same order of magnitude as the unsteady forces.

The attention will be focussed to the starting flow in a slightly viscous fluid, i. e. S and R will be large. Under the additional assumption S and R to be of the same order, one takes in the small parameter

$$\varepsilon=\frac{1}{S}=\frac{1}{\nu' R}.$$

There ν' is a constant of order one. The individuality of both parameters S and R will be retained if one substitutes ε by $1/S$ and ν' by S/R in the solution.

Let (ξ, η) are two-dimensional curvilinear orthogonal co-ordinates such that both Lame's coefficients are equal:

$$h_\xi(\xi, \eta)=h_\eta(\xi, \eta)=h(\xi, \eta).$$

Then the elemental lengths in the ξ and η directions can be written as

$$ds_\xi=h(\xi, \eta)\,d\xi \text{ and } ds_\eta=h(\xi, \eta)\,d\eta, \tag{4}$$

respectively.

In these co-ordinates the equations (2), (3) become

$$\frac{\partial u}{\partial t}+\varepsilon\left(\frac{u}{h}\frac{\partial u}{\partial \xi}+\frac{v}{h}\frac{\partial u}{\partial \eta}+\frac{uv}{h^2}\frac{\partial h}{\partial \eta}-\frac{v^2}{h^2}\frac{\partial h}{\partial \xi}\right)=-\frac{1}{h}\frac{\partial p}{\partial \xi}-\frac{\varepsilon^2\nu'}{h}\frac{\partial}{\partial \eta}\left\{\frac{1}{h^2}\left[\frac{\partial}{\partial \xi}(hv)-\frac{\partial}{\partial \eta}(hu)\right]\right\}, \tag{6}$$

$$\frac{\partial v}{\partial t}+\varepsilon\left(\frac{u}{h}\frac{\partial v}{\partial \xi}+\frac{v}{h}\frac{\partial v}{\partial \eta}+\frac{uv}{h^2}\frac{\partial h}{\partial \xi}-\frac{u^2}{h^2}\frac{\partial h}{\partial \eta}\right)=-\frac{1}{h}\frac{\partial p}{\partial \eta}-\frac{\varepsilon^2\nu'}{h}\frac{\partial}{\partial \xi}\left\{\frac{1}{h^2}\left[\frac{\partial}{\partial \eta}(hu)-\frac{\partial}{\partial \xi}(hv)\right]\right\}, \tag{6}$$

$$\frac{\partial}{\partial \xi}(hu)+\frac{\partial}{\partial \eta}(hv)=0, \tag{7}$$

where u and v are the velocity components in the direction of ξ and η, respectively.

By introduction of a stream function $\psi(\xi, \eta, t)$ as follows:

$$u=\frac{1}{h}\frac{\partial \psi}{\partial \eta}, \quad v=-\frac{1}{h}\frac{\partial \psi}{\partial \xi}, \tag{8}$$

one satisfies the continuity equation.

The object of the consideration will be the cylindrical rigid bodies such that the equation of its cross-section line with the (ξ, η)-plane is $\eta=b$, b being positive. Varying (ξ, η), or respectively $h(\xi, \eta)$, one obtains a class of cylinders which may be obtained one from the other by a conformal mapping.

The initial and boundary conditions are as usual:

$$\psi=0 \text{ for } t\leq 0, \tag{9a}$$

$$\psi\sim\Psi^*(\xi, \eta)\,\Omega(t) \text{ as } t\to 0+ \text{ for all } \eta\geq b, \tag{9}$$

$$\psi=\frac{\partial \psi}{\partial \eta}=0, \text{ or } u=v=0, \text{ on } \eta=b \text{ for } t>0, \tag{10}$$

$$\psi\sim\Psi^*(\xi, \eta)\,\Omega(t) \text{ as } \eta\to\infty \text{ for } t>0 \text{ and all } \xi, \tag{11}$$

where $\Psi^*(\xi, \eta)$ is a stream function of the inviscid potential flow past the cylinder $\eta=b$.

To solve the equations (5)—(7) with respect to the initial and boundary conditions (9)—(11) by the method of matched asymptotic expansions [6, 7] one expresses the variables in series of ε:

in the outer inviscid flow field:

$$\begin{aligned}\psi&=\Psi=\Psi_0+\varepsilon\Psi_1+\cdots,\\ u&=U=U_0+\varepsilon U_1+\cdots,\\ v&=V=V_0+\varepsilon V_1+\cdots,\\ p&=P=P_0+\varepsilon P_1+\cdots;\end{aligned} \tag{12}$$

and in the inner flow field, i. e., in the boundary layer on the cylinder:

$$\eta = b(1+\varepsilon\zeta), \tag{13}$$

$$\psi = \varepsilon\psi_0 + \varepsilon^2\psi_1 + \cdots,$$

$$u = u_0 + \varepsilon u_1 + \cdots, \tag{14}$$

$$v = \varepsilon v_0 + \varepsilon^2 v_1 + \cdots,$$

$$p = p_0 + \varepsilon p_1 + \cdots, \tag{15}$$

$$h(\xi, \eta) = h_0(\xi)[1 + \varepsilon b h_1(\xi)\zeta + \cdots].$$

Here ζ is a new stretched variable, and h_0, $h_0h_1, \ldots$ are the coefficients of a Taylor's expansion of the Lame's coefficient $h(\xi, \eta) = h(\xi, b+\varepsilon b\zeta)$ about $\eta = b$.

Substituting (13)—(15) into (5)—(7), one obtains, for the boundary layer region,

the zeroth-order equations

$$\frac{\partial u_0}{\partial t} - \frac{\nu'}{b^2 h_0^2}\frac{\partial^2 u_0}{\partial \zeta^2} = -\frac{1}{h_0}\frac{\partial p_0}{\partial \xi}, \tag{16}$$

$$\frac{\partial p_0}{\partial \zeta} = 0, \tag{17}$$

and the first-order equations

$$\frac{\partial u_1}{\partial t} - \frac{\nu'}{b^2 h_0^2}\frac{\partial^2 u_1}{\partial \zeta^2} = -\frac{1}{h_0}\frac{\partial p_1}{\partial \xi} + \frac{h_1 b}{h_0}\zeta\frac{\partial p_0}{\partial \xi} - \frac{u_0}{h_0}\frac{\partial u_0}{\partial \xi} - \frac{v_0}{b h_0}\frac{\partial u_0}{\partial \zeta} - 2\zeta\frac{\nu' h_1}{b h_0^2}\frac{\partial^2 u_0}{\partial \zeta^2}, \tag{18}$$

$$\frac{\partial p_1}{\partial \zeta} = 0. \tag{19}$$

Likewise, the equations governing the outer flow result from (12) and (5)—(7). It can be shown [8] from the outer expansion (12) and the initial condition (9a) that in the case of a body motion from rest the external flow is irrotational. Then all the functions Ψ_i $(i = 0, 1, \ldots)$ satisfy the Laplace's equation:

$$\Delta\Psi_i \equiv \frac{\partial^2\Psi_i}{\partial\xi^2} + \frac{\partial^2\Psi_i}{\partial\eta^2} = 0 \quad (i = 0, 1, \ldots). \tag{20}$$

Boundary conditions for the outer flow are these as $\eta \to \infty$ resulting from (11), and for the inner flow the sleep conditions (10) on the wall. Both solutions match in an intermediate region.

Zeroth approximation

One takes

$$\Psi_0(\xi, \eta, t) = \Psi^*(\xi, \eta)\,\Omega(t) \tag{21}$$

as a zeroth approximation to the outer flow stream function. This function satisfies the boundary conditions at infinity.

As the equation (17) shows, the pressure p_0 does not vary across the boundary layer, i. e., $p_0|_{\zeta\to\infty} = P_0|_{\eta=b}$, and then, from the relevant zeroth-order outer flow equation, with respect to $V_0|_{\eta=b} = 0$, results

$$-\frac{1}{h_0}\frac{\partial p_0}{\partial \xi} = \frac{\partial U_0}{\partial t}\bigg|_{\eta=b}. \tag{22}$$

This expression and (16) lead to

(23) $$\frac{\partial \tilde{u}}{\partial t}-\frac{\nu'}{b^2 h_0^2}\frac{\partial^2 \tilde{u}}{\partial \zeta^2}=0,$$

with boundary conditions

(24) $$\tilde{u}|_{\zeta=0}=U_0|_{\eta=b},$$

(25) $$\tilde{u}|_{\zeta\to\infty}=0,$$

where $\tilde{u}=U_0|_{\eta=b}-u_0$.

Introducing a self-similarity variable

(26) $$\eta_1=\frac{\zeta}{2}\frac{bh_0}{\sqrt{\nu' t}}$$

and assuming

$$\tilde{u}=U_0|_{\eta=b}f(\eta_1),$$

one obtains the boundary valueproblem

(27) $$f''(\eta_1)+2\eta_1 f'(\eta_1)-4\alpha f(\eta_1)=0,$$

(28) $$f(0)=1,\quad f(\infty)=0.$$

A solution of (27) is the 2α-th fractional integral of the complementary error function

$$g_\alpha(\eta_1)=\frac{2}{\sqrt{\pi}\,\Gamma(2\alpha+1)}\int_{\eta_1}^{\infty}(\gamma-\eta_1)^{2\alpha}e^{-\gamma^2}d\gamma\quad(\alpha>-1/2).$$

These functions are applied to boundary layer problems by Watson [1], who points at some of their properties:

$$g'_\alpha(\eta_1)=-g_{\alpha-1/2}(\eta_1),\ g_0(\eta_1)=\operatorname{erfc}\eta_1=1-\operatorname{erf}\eta_1,$$

$$g_\alpha(0)=\frac{1}{2^{2\alpha}\Gamma(\alpha+1)},\quad g_\alpha(\infty)=0.$$

The solution of (27), (28)

$$f(\eta_1)=2^{2\alpha}\Gamma(\alpha+1)\,g_\alpha(\eta_1)$$

allows to write

(29) $$u_0=U_0|_{\eta=b}[1-2^{2\alpha}\Gamma(\alpha+1)\,g_\alpha(\eta_1)].$$

From (10), (13)—(15) and (26), one obtains the next relationships between u_0, v_0, u_1, v_1 and ψ_0, ψ_1:

(30) $$\psi_0=2\sqrt{\nu' t}\int_0^{\eta_1}u_0\,d\eta_1,$$

(31) $$v_0=-\frac{1}{h_0}\left(\eta_1\frac{h_0'}{h_0}\frac{\partial}{\partial\eta_1}+\frac{\partial}{\partial\xi}\right)\psi_0,$$

(32) $$\psi_1=2\sqrt{\nu' t}\int_0^{\eta_1}u_1\,d\eta_1+\frac{4h_1}{h_0}\nu' t\int_0^{\eta_1}\eta_1 u_0\,d\eta_1,$$

(33) $$v_1=-\frac{1}{h_0}\left(\eta_1\frac{h_0'}{h_0}\frac{\partial}{\partial\eta_1}+\frac{\partial}{\partial\xi}\right)\psi_1+2\sqrt{\nu' t}\,\frac{h_1}{h_0^2}\,\eta_1\left(\eta_1\frac{h_0'}{h_0}\frac{\partial}{\partial\eta_1}+\frac{\partial}{\partial\xi}\right)\psi_0,$$

where $(\,.\,)'$ denotes differentiation with respect to ξ. The formulas (29) and (30) give for ψ_0

(34) $$\psi_0=2\sqrt{\nu' t}\,U_0|_{\eta=b}\left[\eta_1-\frac{\Gamma(\alpha+1)}{2\Gamma(\alpha+3/2)}+2^{2\alpha}\Gamma(\alpha+1)\,g_{\alpha+1/2}(\eta_1)\right],$$

and from (31) and (34) yields

(35) $$v_0=-\frac{2\sqrt{\nu' t}}{h_0}\left\{U_0'|_{\eta=b}\left[\eta_1-\frac{\Gamma(\alpha+1)}{2\Gamma(\alpha+3/2)}+2^{2\alpha}\,\Gamma(\alpha+1)g_{\alpha+1/2}(\eta_1)\right]\right.$$
$$\left.+\frac{h_0'}{h_0}U_0|_{\eta=b}[\eta_1-2^{2\alpha}\Gamma(\alpha+1)\,\eta_1 g_\alpha(\eta_1)]\right\}.$$

First approximation

In the right-hand side of the equation (18) all the terms, with the exception of $-\frac{1}{h_0}\frac{\partial p_1}{\partial\xi}$, may be regarded as known. The second equation (19) permits this term to be evaluated, provided $U_1|_{\eta=b}$ is known. Indeed, from (19) it follows

$$p_1(\xi,\zeta)=P_1{}_{\eta=b},\ \text{i. e.,}\ -\frac{1}{h_0}\frac{\partial p_1}{\partial\xi}=-\frac{1}{h_0}\frac{\partial P_1}{\partial\xi}\bigg|_{\eta=b},$$

which with respect to the first approximation to the respective outer flow equation and $V_0|_{\eta=b}=0$ yields

(36) $$-\frac{1}{h_0}\frac{\partial p_1}{\partial\xi}=\frac{\partial}{\partial t}(U_1|_{\eta=b})+\frac{1}{h_0}U_0|_{\eta=b}\cdot U_0'|_{\eta=b}.$$

Since $U_1=\frac{1}{h}\frac{\partial\Psi_1}{\partial\eta}$, it is necessary to be found Ψ_1. This function is determined by the equation (20), the condition

(37) $$\frac{\partial\Psi_1}{\partial\eta}\bigg|_{\eta\to\infty}=0\ \ (\text{or}\ U_1|_{\eta\to\infty}=0),$$

and the requirement $\Psi_0+\varepsilon\Psi_1$ to match $\varepsilon\psi_1$ in an overlap region. The latter leads to [6, 7]

(38) $$\Psi_1|_{\eta=b}=-\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+3/2)}\sqrt{\nu' t}\,U_0|_{\eta=b}.$$

The solution of the equation (20), subject to the boundary conditions (37), (38), for $|\xi|<l(l>0)$, is

(39) $$\Psi_1=-\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+3/2)}\sqrt{\nu' t}\,\Omega(t)\left[\frac{a_0}{2}+\sum_{k=1}^{\infty}e^{-\frac{k\pi}{l}(\eta-b)}\left(a_k\cos\frac{k\pi}{l}\xi+b_k\sin\frac{k\pi}{l}\xi\right)\right],$$

where

(40) $$a_k=\frac{1}{l}\int_{-l}^{l}W(\xi)\cos\frac{k\pi}{l}\xi\,d\xi\qquad(k=0,1,\ldots),$$

(41) $$b_k=\frac{1}{l}\int_{-l}^{l} W(\xi)\sin\frac{k\pi}{l}\xi\,d\xi \qquad (k=1,2,\ldots)$$

are the Fourier's coefficients of the function

(42) $$W(\xi)=\frac{U_0|_{\eta=b}}{\Omega(t)}=\frac{1}{h_0}\frac{\partial\Psi^*}{\partial\eta}\Big|_{\eta=b}.$$

The relation

(43) $$U_1|_{\eta=b}=\frac{1}{h_0}\frac{\partial\Psi_1}{\partial\eta}\Big|_{\eta=b}$$

and the substitution

(44) $$F(\xi)=(\sqrt{\nu' t}\,\Omega(t))^{-1}U_1|_{\eta=b}$$

lead to

(45) $$\frac{\partial}{\partial t}(U_1|_{\eta=b})=\sqrt{\frac{\nu'}{t}}\,\Omega(t)\left(\alpha+\frac{1}{2}\right)F(\xi).$$

Then from (18) with respect to (29), (35), (22), (36), (42) — (45) and (26) one obtains the following equation for the first approximation:

(46) $$\frac{\partial^2 u_1}{\partial\eta_1^2}+2\eta_1\frac{\partial u_1}{\partial\eta_1}-4t\frac{\partial u_1}{\partial t}=-4\sqrt{\nu' t}\,\Omega(t)\,G(\eta_1)-\frac{4}{h_0}t\Omega^2(t)W(\xi)W'(\xi)H(\eta_1),$$

where

(47) $$G(\eta_1)=(\alpha+1/2)F(\xi)-\frac{h_1}{h_0}W(\xi)[2\alpha\eta_1-2^{2\alpha}\Gamma(\alpha+1)\eta_1 g_{\alpha-1}(\eta_1)],$$

(48) $$H(\eta_1)=2^{2\alpha+1}\Gamma(\alpha+1)g_\alpha(\eta_1)-2^{4\alpha}\Gamma^2(\alpha+1)g_\alpha^2(\eta_1)$$

$$+2^{2\alpha}\Gamma(\alpha+1)\left[\eta_1 g_{\alpha-1/2}(\eta_1)-\frac{\Gamma(\alpha+1)}{2\Gamma(\alpha+3/2)}g_{\alpha-1/2}(\eta_1)+2^{2\alpha}\Gamma(\alpha+1)\cdot g_{\alpha-1/2}(\eta_1)g_{\alpha+1/2}(\eta_1)\right.$$

Substituting

(49) $$u_1=-4\sqrt{\nu' t}\,\Omega(t)\,X(\eta_1)-\frac{4}{h_0}t\Omega^2(t)W(\xi)W'(\xi)\,Y(\eta_1)$$

into (46), it follows for $X(\eta_1)$ and $Y(\eta_1)$:

(50) $$X''(\eta_1)+2\eta_1 X'(\eta_1)-4(\alpha+1/2)X(\eta_1)=G(\eta_1),$$

(51) $$Y''(\eta_1)+2\eta_1 Y'(\eta_1)-4(2\alpha+1)Y(\eta_1)=H(\eta_1).$$

From the sleep condition $u_1|_{\eta_1=0}=0$, the boundary conditions on $\eta_1=0$ are

(52) $$X(0)=0, \quad Y(0)=0.$$

The conditions for $X(\eta_1)$ and $Y(\eta_1)$ as $\eta_1\to\infty$ follow from the requirement $u_0+\varepsilon u_1$ to match $U_0+\varepsilon U_1$ in an overlap region. The solutions of (50), (51) satisfying (52) for which the matching procedure in question is possible are

(53) $$X(\eta_1)=2^{2\alpha-2}\left[2\Gamma(\alpha+3/2)F(\xi)-\frac{h_1}{h_0}W(\xi)\,\Gamma(\alpha+1)(2\alpha-1)(2\alpha+1)\right]g_{\alpha+1/2}(\eta_1)$$

$$-\frac{1}{4}F(\xi)+\frac{h_1}{2h_0}W(\xi)\eta_1+\frac{h_1}{h_0}W(\xi)2^{2\alpha-4}\Gamma(\alpha+1)[-g_{\alpha-3/2}(\eta_1)+4(2\alpha-1)g_{\alpha-1/2}(\eta_1)],$$

(54) $$Y(\eta_1)=-2^{4\alpha-1}\Gamma(2\alpha+2)\left[\frac{\alpha-6}{(\alpha+1)(\alpha+2)}-\frac{\Gamma^2(\alpha+1)}{\Gamma^2(\alpha+3/2)}+\frac{2\Gamma^2(\alpha+1)}{\Gamma(\alpha+5/2)\Gamma(\alpha+1/2)}\right]g_{2\alpha+1}(\eta_1)$$

$$+2^{2\alpha-3}\Gamma(\alpha+1)\left[-\frac{1}{\alpha+2}g_{\alpha-1}(\eta_1)+\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+5/2)}g_{\alpha-1/2}(\eta_1)+\frac{4(\alpha-1)}{\alpha+1}g_{\alpha}(\eta_1)\right]$$
$$+2^{4\alpha-1}\Gamma^2(\alpha+1)[g_{\alpha}(\eta_1)g_{\alpha+1}(\eta_1)-g^2_{\alpha+1/2}(\eta_1)].$$

Indeed, introducing an intermediate variable

(55) $$\bar{\eta}_1=\sqrt{\varepsilon}\eta_1,$$

one obtains

(56) $$U_0+\varepsilon U_1=U_0|_{\eta=b}+\left.\frac{\partial U_0}{\partial\eta}\right|_{\eta=b}(\eta-b)+\cdots+\varepsilon(U_1|_{\eta=b}+\cdots)$$
$$=U_0|_{\eta=b}+\left.\frac{\partial U_0}{\partial\eta}\right|_{\eta=b}\cdot\frac{2\sqrt{\nu' t}}{h_0}\sqrt{\varepsilon}\,\bar{\eta}_1+\varepsilon U_1|_{\eta=b}+O(\varepsilon\sqrt{\varepsilon}\,\bar{\eta}_1,\ \varepsilon\bar{\eta}_1^2),$$

(57) $$u_0+\varepsilon u_1=U_0|_{\eta=b}+\varepsilon U_1|_{\eta=b}-4\sqrt{\nu' t}\,\Omega(t)\frac{h_1}{2h_0}W(\xi)\sqrt{\varepsilon}\,\bar{\eta}_1$$
$$+\text{ transc. small terms.}$$

The comparison of both expresions (56) and (57) shows that the matching is possible, only provided

$$\left.\frac{\partial U_0}{\partial\eta}\right|_{\eta=b}=-h_1\Omega(t)\,W(\xi),$$

i. e. with respect to (42), provided

(58) $$\left.\frac{\partial U_0}{\partial\eta}\right|_{\eta=b}=-h_1U_0|_{\eta=b}.$$

By definition $U_0=\frac{1}{h}\frac{\partial\Psi_0}{\partial\eta}$. Then

$$\frac{\partial U_0}{\partial\eta}=-\frac{\frac{\partial h}{\partial\eta}}{h^2}\frac{\partial\Psi_0}{\partial\eta}+\frac{1}{h}\frac{\partial^2\Psi_0}{\partial\eta^2},$$

and on $\eta=b$

(59) $$\left.\frac{\partial U_0}{\partial\eta}\right|_{\eta=b}=-\frac{\left.\frac{\partial h}{\partial\eta}\right|_{\eta=b}}{h^2|_{\eta=b}}\left.\frac{\partial\Psi_0}{\partial\eta}\right|_{\eta=b}+\frac{1}{h|_{\eta=b}}\left.\frac{\partial^2\Psi_0}{\partial\eta^2}\right|_{\eta=b}.$$

But $\frac{\partial^2\Psi_0}{\partial\xi^2}+\frac{\partial^2\Psi_0}{\partial\eta^2}=0$, hence

(60) $$\left.\frac{\partial^2\Psi_0}{\partial\eta^2}\right|_{\eta=b}=-\frac{\partial^2}{\partial\xi^2}(\Psi_0|_{\eta=b}).$$

In addition, because $\eta=b$ is the zeroth stream line $\Psi_0|_{\eta=b}=0$, it follows from (60):

(61) $$\left.\frac{\partial^2\Psi_0}{\partial\eta^2}\right|_{\eta=b}=0.$$

Finally, one results from (15)

(72) $$h|_{\eta=b}=h_0(\xi),\quad\left.\frac{\partial h}{\partial\eta}\right|_{\eta=b}=h_0(\xi)h_1(\xi).$$

Then, substituting (61), (62) into (59), one obtains the required (58).

The formulas (49), (53) and (54) give u_1. It is necessary to be noted, that the first term in the expression (49) for u_1 is absent in the Watson's second approximation.

The stream function ψ_1 and the velocity component v_1 are given by (32) and (33).

A uniformly valid solution can be constructed by the well known technique [6, 7] from the evaluated outer and inner expansions of the flow field variables. The method is systematic and higher approximation can be found.

Zero shear stress points

The separation (reattachment) points are these of zero shear stress. To the second order they are given by

$$\left(\frac{\partial u_0}{\partial \eta_1}+\varepsilon\frac{\partial u_1}{\partial \eta_1}\right)\Big|_{\eta_1=0}=0. \tag{63}$$

Introducing a non-dimensional time T as a ratio of the actual time $t'=T_0t$ to the time which takes to travel the characteristic length L_0 with the characteristic velocity U_0, i. e.,

$$T=\frac{t'}{L_0/U_0}=\frac{tT_0U_0}{L_0}=\frac{t}{S}=\varepsilon t, \tag{64}$$

one obtains from (63), (64), (29) and (49) for the time and the position of separation (reattachment):

$$\frac{2}{h_0}W(\xi)W'(\xi)Y'(0)(\nu'R)^{\alpha}AT^{\alpha+1}+2\frac{X'(0)}{\sqrt{R}}\sqrt{T}-\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1/2)}W(\xi)=0. \tag{65}$$

In terms of the dimensionless distance σ, defined as a ratio between the distance travelled by the cylinder and the characteristic length L_0, i. e.

$$\sigma=\frac{\int_0^{t'}U_0At^{\alpha}\,dt'}{L_0}=\frac{A}{\alpha+1}(\nu'R)^{\alpha}T^{\alpha+1}, \tag{66}$$

the last equation becomes

$$\frac{2}{h_0}W(\xi)W'(\xi)Y'(0)(\alpha+1)\sigma+2X'(0)\frac{1}{\sqrt{R}}\left[\frac{(\alpha+1)\sigma}{A(\nu'R)^{\alpha}}\right]^{\frac{1}{2(\alpha+1)}}-\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1/2)}W(\xi)=0. \tag{67}$$

In the limit of infinite Reynolds number, one obtains from (67)

$$\sigma_{\infty}=\Gamma(\alpha+1)\left[(\alpha+1)\,\Gamma(\alpha+1/2)\,\frac{2}{h_0}\,W'(\xi)\,Y'(0)\right]^{-1}. \tag{68}$$

It can be shown with respect to (4) that σ_{∞} does not differ from this given by Watson's (66) and (68) [1]. Also, the additional term (that with $X'(0)$) which appears in the expression for the friction of the present analysis and which is due to the displacement effect of the boundary layer on the outer flow is essential only for finite Reynolds numbers.

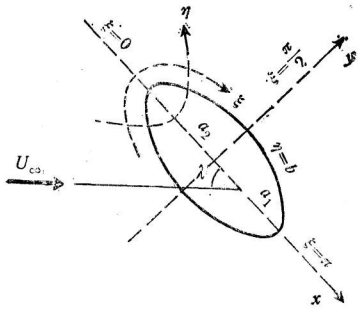


Fig. 1. Co-ordinate system

Elliptic cylinder case

If one introduces elliptic co-ordinates (Fig. 1) such that

$$x+iy=2e^{-b}\operatorname{ch}[\eta+i(\pi-\xi)],$$

$$0\leq\xi\leq 2\pi,\quad \eta\geq b>0,$$

. e.

$$x=-2e^{-b}\operatorname{ch}\eta\cos\xi,\tag{69}$$

$$y=2e^{-b}\operatorname{sh}\eta\sin\xi,$$

$$h(\xi,\eta)=2e^{-b}(\operatorname{sh}^2\eta+\sin^2\xi)^{1/2},\tag{70}$$

x, y being the Cartesian co-ordinates, the equation $\eta=b$ describes the surface of an elliptic cylinder with half-axis $a_1=2e^{-b}\operatorname{ch}b$, $a_2=2e^{-b}\operatorname{sh}b$. One obtains from [9] the following stream function for the potential flow past this cylinder with a velocity at infinity U_∞ at an angle of attack λ:

$$\Psi^*(\xi,\eta)=2\sin(\lambda+\xi)\operatorname{sh}(\eta-b).\tag{71}$$

In (71) the velocities are nondimentionalized by U_∞ and the lengths by $(a_1+a_2)/2$.

The case $\alpha=0$ is that discussed by Wang [5].

The movement of the zero shear stress points in the case of a uniformly accelerated ($\alpha=1$) 2:1 elliptic cylinder is presented in Fig. 2 for $R=100$ and for various angles of attack. The curves show the distance of separation (or reattachment) σ for each position ξ on the cylinder. For example, let $\lambda=\pi/4$. The first separation takes a place at $\xi=2.93$ when $\sigma=0.261$ (point A). The bubble which is formed in this point becomes larger as the travelled distance increases. In so doing, the rear part of the bubble moves more slowly than the forward part as indicates the slope of the curve. When $\sigma=0.996$ at $\xi=0.58$, another bubble occurs (point B). The forward part of the first bubble and the rear part of the second bubble move quickly one to the other and two bubbles coalesce into a large wake when $\sigma=1.523$ at $\xi=1.43$ (point C). Following the Wang's terminology [5], the distances travelled by the cylinder from fhe starting to the positions corresponding to the points A, B and C will be called the first and second separation distances and the stall distance, respectively.

The effect of the finite Reynolds numbers on the movement of the separation points can be seen in Fig. 3, where two curves are plotted, that for $R=\infty$ and that for $R=10$. The greatest relative difference between them is ot the order of 8 percent. The dependence of the first and the second separation distances and of the stall distance on Reynolds number is presented in Fig. 4.

Fig. 5 shows the separation distances and the stall distance versus α for $R=100$. The dependence on α is weakest for the first separation distance and strongest for the stall distance. All the curves as $\alpha\to\infty$ asymptotically approach the respective values for exponentially accelerated body (Watson [1]).

The separation distance as a function of the axis ratio $k=a_1/a_2$ is plotted in Fig. 6 for the symmetrical flow streaming past an elliptic cylinder and for $\alpha=1$, $R=100$. For k larger than $\sqrt{3}/2$ approximately, the first separation occurs in the rear stagnation point. When $k=\sqrt{3}/2$ is reached, the separation begins

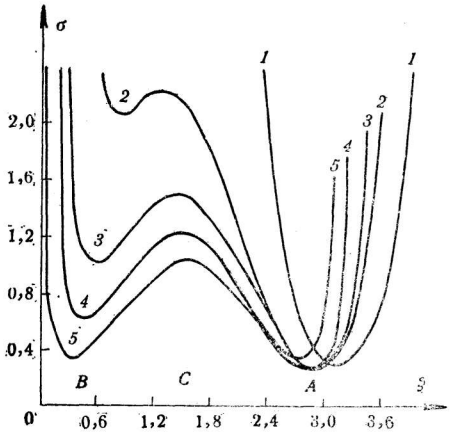


Fig. 2. Movement of zero shear stress points for a uniformly accelerated 2 : 1 elliptic cylinder at various angles of attack, $R=100$:

1 — $\lambda=0$, *2* — $\lambda=\pi/6$, *3* — $\lambda=\pi/4$, *4* — $\lambda=\pi/3$, *5* — $\lambda=\pi/2$; *A* — first separation point, *B* — second separation point, *C* — stall point

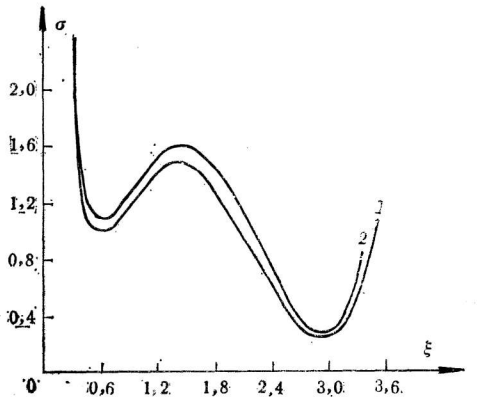


Fig. 3. Movement of zero shear stress points for a uniformly accelerated 2 : 1 elliptic cylinder at angle of attack

$\lambda=\pi/4$: *1* — $R=\infty$ (Watson), *2* — $R=10$

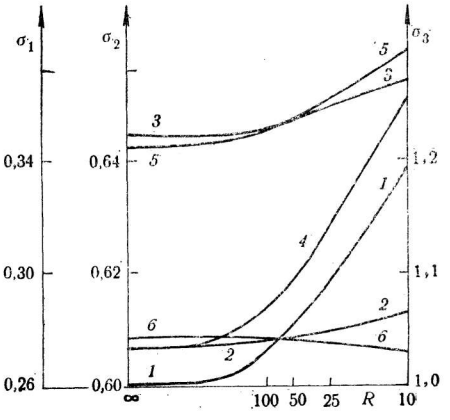


Fig. 4. Effect of Reynolds number on separation distances and stall distance for a 2 : 1 elliptic cylinder, $\alpha=1$: σ_1 — first separation distance:

1 — $\lambda=0$, *2* — $\lambda=\pi/3$, *3* — $\lambda=\pi/2$; σ_2 — second separation distance: *4* — $\lambda=\pi/3$; σ_3 — stall distance: *5* — $\lambda=\pi/3$, *6* — $\lambda=\pi/2$

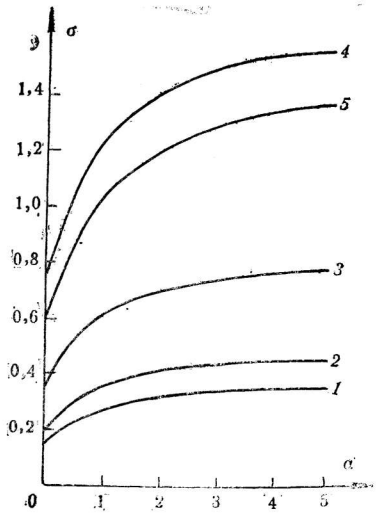


Fig. 5. Acceleration effect on separation distances and stall distance for a 2 : 1 elliptic cylinder, $R=100$: first separation distance:

1 — $\lambda=\pi/3$, *2* — $\lambda=\pi/2$; second separation distance: *3* — $\lambda=\pi/3$; stall distance: *4* — $\lambda=\pi/3$, *5* — $\lambda=\pi/2$

at two points and the (k, σ)-curve bifurcates in the point A which is slightly different from the maximum point. The value for $k=1$ is taken from a similar analysis given by Slavchev [10]. For comparison, the results of Ashkovich [11] obtained by Blasius' boundary layer method [2] are presented also.

All the computations shown in the figures are made for $A=\nu'=1$.

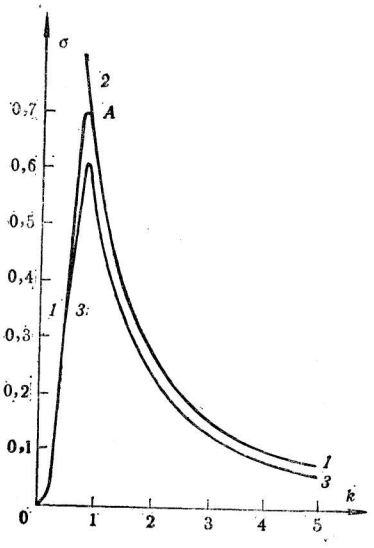


Fig. 6. Axis ratio dependence of separation distance for symmetrical flow streaming past an elliptic cylinder, $\alpha=1$:
1 — first separation distance, $R=100$; *2* — stall distance, $R=100$; *3* — first separation distance, $R=\infty$ (Blasius, Ashkovich); *A* — two separation points appearence point

Other special cases

The case $h(\xi, \eta)=\sqrt{\xi^2+\eta^2}$, $\Psi^*(\xi, \eta)=\xi(\eta-1)$ is that of the symmetrical flow streaming past the parabolic cylinder $\eta=1$ investigated in [12] for $\alpha=0$ and in [8] for $\alpha\geq 0$. Unfortunately, some terms are absent in the second approximation of the latter.

The case $h(\xi, \eta)=\sqrt{\xi^2+\eta^2}$, $\Psi^*(\xi, \eta)=(\xi+k)(\eta-1)$ of the flow about the same cylinder at an angle of attack defined by k is considered by Zapryanov and Kalitzova-Kurteva [13].

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