

Thermoviscoelasticity with a Temperature Rate Dependence

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1. Introduction

It was observed by many authors that some difficulties arise if the rate of temperature change is included as an independent variable in the constitutive equations. Bogy and Naghdi [1] have shown that constitutive assumptions of this type for rigid solids led to finite speed of heat propagation in certain circumstances, but not for a theory linearized about a uniform temperature. On the other hand, it is well-known that if the temperature rate is excluded then the heat conduction equation is parabolic and entails an infinite speed of heat propagation. In order to avoid this difficulties Müller [2, 3], Green and Laws [4] have proposed alternative generalizations of the entropy production inequality that lead to finite speed of heat propagation in a rigid solid when the corresponding heat conduction equations are linearized. Similar results are also obtained by Green, Lindsay [5] and Green [6] for thermoelastic solids.

The non-linear theory for thermoviscoelastic Kelvin-Voigt solids when the temperature rate is excluded from the constitutive equations was given by Koh and Eringen [7]. In the present paper a generalization of this theory is considered because of two reasons. We think that it is more natural to have a stress-strain relation that reduces in the one-dimensional linear case to the form

$$\tau = E(e - \alpha_T T) + E' \frac{d}{dt} (e - \alpha'_T T),$$

where τ is the tension in a bar, e is the strain, T is the temperature change, E is the Young's modulus, E' is the viscosity, α_T is the coefficient of linear thermal expansion and α'_T is a material constant connected with the temperature rate. When $\alpha'_T = 0$ we obtain the classical stress-strain relation. When $\alpha'_T = \alpha_T$ we have that the tension depends linearly on the elastic part of deformation and its time derivative. We shall also suppose that the linearized heat conduction equation uncoupled with the strain must be a hyperbolic differential equation and therefore will predict a finite speed of heat propagation. The proposed generalization concerns with non-linear and homogeneous thermoviscoelastic Kelvin-Voigt solids when a dependence on the temperature rate is assumed. It is based on the entropy production inequality suggested by Green and Laws [4].

2. Notation and Field Equations

Let the motion of the continuum be referred to a fixed system of rectangular Cartesian axes and the position of a typical particle be designated by the equations

$$(2.1) \quad x_i = x_i(X_A, t),$$

where X_A are the co-ordinates in the reference state and x_i are the spatial co-ordinates at time t . Latin indices take values 1, 2, 3 and the usual summation convention will be employed. The deformation gradient with respect to the reference state is given by

$$(2.2) \quad x_{i,A} = \frac{\partial}{\partial X_A} x_i(X_A, t)$$

and the Jacobian

$$(2.3) \quad j = \det x_{i,A}$$

is assumed positive. A superposed dot denotes the material time derivative and, in particular,

$$(2.4) \quad v_i \equiv \dot{x}_i = \frac{\partial}{\partial t} x_i(X_A, t).$$

The following balance equations are valid:

(i) Equation of balance of mass

$$(2.5) \quad \dot{\rho}_0 = j\dot{\rho},$$

where $\rho_0(>0)$ is the reference mass density and ρ is the mass density at time t .

(ii) Equation of balance of linear momentum

$$(2.6) \quad (T_{AB}x_{i,B})_{,A} + \rho_0 F_i = \rho_0 \dot{v}_i,$$

where T_{AB} is the Piola-Kirchoff stress tensor and F_i is the specific externally applied body force.

(iii) Equation of balance of moment of momentum

$$(2.7) \quad T_{AB} = T_{BA}.$$

(iv) Equation of balance of energy

$$(2.8) \quad \rho_0 \dot{\varepsilon} = T_{AB} \dot{E}_{AB} - Q_{A,A} + \rho_0 r,$$

where ε is the specific internal energy,

$$(2.9) \quad 2E_{AB} = x_{i,A} x_{i,B} - \delta_{AB}$$

is the strain tensor, δ_{AB} is the Kronecker symbol, Q_A is the heat flux vector per unit area of the X_A -plane, but acting over the corresponding surface at time t and r is the externally supplied specific heat.

We adopt the entropy production inequality that was suggested by Green and Laws [4], namely

$$(2.10) \quad \frac{d}{dt} \int_P \rho_0 \eta dV - \int_P \frac{\rho_0 r}{\Phi} dV + \int_B \frac{Q_A}{\Phi} dS_A \geq 0.$$

It holds for every volume P bounded by a closed surface B in the reference configuration, where η is the specific entropy and Φ is a scalar function. It is supposed that $\Phi > 0$ and in thermostatics Φ reduces to the absolute temperature $\theta(>0)$. From (2.10) and (2.8) we obtain the following entropy production inequality

$$(2.11) \quad T_{AB} \dot{E}_{AB} - \varrho_0 (\dot{\psi} + \eta \dot{\Phi}) - \frac{Q_A \Phi_{,A}}{\Phi} \geq 0,$$

where

$$(2.12) \quad \psi = \varepsilon - \eta \Phi.$$

3. Constitutive Equations

We consider a non-linear and homogeneous thermoviscoelastic Kelvin-Voigt solid when a dependence on the temperature rate is assumed. Then the functions

$$(3.1) \quad \psi, \Phi, \eta, Q_A, T_{AB}$$

are functions of the variables

$$(3.2) \quad x_{i,A}, \dot{x}_{i,A}, \theta, \dot{\theta}, \theta_{,A}.$$

According to the invariance conditions under superposed rigid body motion we must replace the variables in (3.2) by

$$(3.3) \quad E_{AB}, \dot{E}_{AB}, \theta, \dot{\theta}, \theta_{,A}.$$

Bearing in mind (3.3) we obtain from (2.11) the following inequality

$$(3.4) \quad \left[T_{AB} - \frac{1}{2} \varrho_0 \left(\frac{\partial \psi}{\partial E_{AB}} + \frac{\partial \psi}{\partial E_{BA}} \right) - \frac{1}{2} \varrho_0 \eta \left(\frac{\partial \Phi}{\partial E_{AB}} + \frac{\partial \Phi}{\partial E_{BA}} \right) \right] \dot{E}_{AB} \\ - \frac{1}{2} \varrho_0 \left[\frac{\partial \psi}{\partial \dot{E}_{AB}} + \frac{\partial \psi}{\partial \dot{E}_{BA}} + \eta \left(\frac{\partial \Phi}{\partial \dot{E}_{AB}} + \frac{\partial \Phi}{\partial \dot{E}_{BA}} \right) \right] \ddot{E}_{AB} - \varrho_0 \left(\frac{\partial \psi}{\partial \theta} + \eta \frac{\partial \Phi}{\partial \theta} \right) \dot{\theta} \\ - \varrho_0 \left(\frac{\partial \psi}{\partial \dot{\theta}} + \eta \frac{\partial \Phi}{\partial \dot{\theta}} \right) \ddot{\theta} - \varrho_0 \left(\frac{\partial \psi}{\partial \theta_{,A}} + \eta \frac{\partial \Phi}{\partial \theta_{,A}} \right) \dot{\theta}_{,A} - \frac{Q_A}{\Phi} \left[\frac{1}{2} \left(\frac{\partial \Phi}{\partial E_{BC}} + \frac{\partial \Phi}{\partial E_{CB}} \right) E_{BC,A} \right. \\ \left. + \frac{1}{2} \left(\frac{\partial \Phi}{\partial \dot{E}_{BC}} + \frac{\partial \Phi}{\partial \dot{E}_{CB}} \right) \dot{E}_{BC,A} + \frac{\partial \Phi}{\partial \theta} \theta_{,A} + \frac{\partial \Phi}{\partial \dot{\theta}} \dot{\theta}_{,A} + \frac{\partial \Phi}{\partial \theta_{,B}} \theta_{,BA} \right] \geq 0.$$

This inequality must be satisfied for all deformation and temperature fields since the energy and momentum equations can be balanced by suitable choice of r and F_i . Then we have

$$(3.5) \quad \frac{\partial \psi}{\partial \dot{E}_{AB}} + \frac{\partial \psi}{\partial \dot{E}_{BA}} + \eta \left(\frac{\partial \Phi}{\partial \dot{E}_{AB}} + \frac{\partial \Phi}{\partial \dot{E}_{BA}} \right) = 0,$$

$$(3.6) \quad \frac{\partial \psi}{\partial \dot{\theta}} + \eta \frac{\partial \Phi}{\partial \dot{\theta}} = 0,$$

$$(3.7) \quad \varrho_0 \left(\frac{\partial \psi}{\partial \theta_{,A}} + \eta \frac{\partial \Phi}{\partial \theta_{,A}} \right) + \frac{Q_A}{\Phi} \frac{\partial \Phi}{\partial \theta} = 0,$$

$$(3.8) \quad \frac{\partial \Phi}{\partial E_{BC}} + \frac{\partial \Phi}{\partial E_{CB}} = 0,$$

$$(3.9) \quad \frac{\partial \Phi}{\partial \dot{E}_{BC}} + \frac{\partial \Phi}{\partial \dot{E}_{CB}} = 0,$$

$$(3.10) \quad Q_A \frac{\partial \Phi}{\partial \theta_{,B}} + Q_B \frac{\partial \Phi}{\partial \theta_{,A}} = 0,$$

$$(3.11) \quad \left[T_{AB} - \frac{1}{2} \varrho_0 \left(\frac{\partial \psi}{\partial E_{AB}} + \frac{\partial \psi}{\partial E_{BA}} \right) - \frac{1}{2} \varrho_0 \eta \left(\frac{\partial \Phi}{\partial E_{AB}} + \frac{\partial \Phi}{\partial E_{BA}} \right) \right] \dot{E}_{AB} - \varrho_0 \left(\frac{\partial \psi}{\partial \theta} + \eta \frac{\partial \Phi}{\partial \theta} \right) \dot{\theta} - \frac{Q_A}{\Phi} \frac{\partial \Phi}{\partial \theta} \theta_{,A} \geq 0,$$

provided $Q_A Q_{,A} \neq 0$. From (3.8)–(3.10) it follows that Φ does not depend on E_{AB} , $\dot{E}_{AB}$ and $\theta_{,A}$. From (3.5) we have that ψ does not depend on $\dot{E}_{AB}$.

We introduce the notation

$$(3.12) \quad \Sigma = \varrho_0 \psi$$

and decompose T_{AB} in two parts:

$$(3.13) \quad T_{AB} = T_{AB}^E + T_{AB}^D,$$

where

$$(3.14) \quad T_{AB}^E = T_{BA}^E = \frac{1}{2} \left(\frac{\partial \Sigma}{\partial E_{AB}} + \frac{\partial \Sigma}{\partial E_{BA}} \right).$$

From (2.7) it follows that $T_{AB}^D = T_{BA}^D$. Then (3.11) reduces to

$$(3.15) \quad T_{AB}^D \dot{E}_{AB} - \chi \dot{\theta} - \frac{Q_A}{\Phi} \frac{\partial \Phi}{\partial \theta} \theta_{,A} \geq 0$$

with

$$(3.16) \quad \chi = \frac{\partial \Sigma}{\partial \theta} + \varrho_0 \eta \frac{\partial \Phi}{\partial \theta}.$$

The inequality (3.15), in particular, implies that

$$(3.17) \quad T_{AB}^D |_{\dot{E}_{AB}=0, \dot{\theta}=0, \theta_{,A}=0} = 0,$$

$$(3.18) \quad \chi |_{\dot{E}_{AB}=0, \dot{\theta}=0, \theta_{,A}=0} = 0,$$

$$(3.19) \quad Q_A |_{\dot{E}_{AB}=0, \dot{\theta}=0, \theta_{,A}=0} = 0,$$

provided T_{AB}^D , χ and Q_A are continuous in $\dot{E}_{AB}$, $\dot{\theta}$ and $\theta_{,A}$.

Supposing that $\frac{\partial \Phi}{\partial \theta} \neq 0$ we obtain from (3.6) and (3.7) as follows:

$$(3.20) \quad \varrho_0 \eta = - \frac{\frac{\partial \Sigma}{\partial \theta}}{\frac{\partial \Phi}{\partial \theta}},$$

$$(3.21) \quad Q_A = - \Phi \frac{\frac{\partial \Sigma}{\partial \theta_{,A}}}{\frac{\partial \Phi}{\partial \theta}}.$$

With the help of (2.12), (3.12)–(3.14), (3.20) and (3.21), the energy equation (2.8) reduces to

$$(3.22) \quad \varrho_0 r - Q_{A,A} + T_{AB}^D \dot{E}_{AB} - \varrho_0 \dot{\eta} \Phi - \left(\frac{\partial \Sigma}{\partial \theta} + \varrho_0 \eta \frac{\partial \Phi}{\partial \theta} \right) \dot{\theta} - \frac{\partial \Sigma}{\partial \theta_{,A}} \dot{\theta}_{,A} = 0.$$

4. Linear Constitutive Equations

For the linear theory it is appropriate to employ the infinitesimal strain tensor

$$(4.1) \quad e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}),$$

where u_i is the displacement and $,i$ denotes the partial derivative with respect to x_i . We now use x_i as co-ordinates of the points in the reference configuration and write t_{ij} , t_{ij}^E , t_{ij}^D , q_i and σ instead of T_{AB} , T_{AB}^E , T_{AB}^D , Q_A and Σ .

A linear theory with respect to $\dot{e}_{ij}$, $\dot{\theta}$ and $\theta_{,i}$ requires

$$(4.2) \quad \sigma = \sigma^{(0)} + \sigma^{(1)} \dot{\theta} + \sigma_i^{(1)} \theta_{,i} + \frac{1}{2} \sigma^{(2)} \dot{\theta}^2 + \sigma_i^{(2)} \dot{\theta} \theta_{,i} + \frac{1}{2} \sigma_{ij}^{(2)} \theta_{,i} \theta_{,j};$$

$$(4.3) \quad \Phi = \Phi^{(0)} + \Phi^{(1)} \dot{\theta} + \frac{1}{2} \Phi^{(2)} \dot{\theta}^2,$$

$$(4.4) \quad t_{ij}^D = \kappa^{(0)} + \kappa_{ijkl}^{(1)} \dot{e}_{kl} + \kappa_{ij}^{(1)} \dot{\theta} + \kappa_{ijk}^{(1)} \theta_{,k},$$

where the coefficients in the expansions of σ and t_{ij}^D depend on e_{ij} , θ and in Φ on θ only. We also have

$$(4.5) \quad \sigma_{ij}^{(2)} = \sigma_{ji}^{(2)}, \quad \kappa_{ijkl}^{(1)} = \kappa_{jikl}^{(1)} = \kappa_{ijlk}^{(1)}, \quad \kappa_{ij}^{(1)} = \kappa_{ji}^{(1)}, \quad \kappa_{ijk}^{(1)} = \kappa_{jik}^{(1)}.$$

If we assume that the part of stress t_{ij}^D that does not contribute to dissipation is zero then

$$(4.6) \quad \kappa_{ijkl}^{(1)} = \kappa_{klij}^{(1)}.$$

Since Φ reduces to θ in thermostatics we have

$$(4.7) \quad \Phi|_{\dot{\theta}=0} = \Phi^{(0)} = \theta.$$

Substituting (4.2)–(4.4) into (3.20) and (3.21) we obtain after linearization

$$(4.8) \quad \varrho_0 \eta = -\frac{1}{\Phi^{(1)}} \left[\sigma^{(1)} + \left(\sigma^{(2)} - \frac{\Phi^{(2)}}{\Phi^{(1)}} \sigma^{(1)} \right) \dot{\theta} + \sigma_i^{(2)} \theta_{,i} \right],$$

$$(4.9) \quad q_i = -\frac{1}{\Phi^{(1)}} \left[\theta \sigma_i^{(1)} + \left(\Phi^{(1)} \sigma_i^{(1)} - \theta \frac{\Phi^{(2)}}{\Phi^{(1)}} \sigma_i^{(1)} + \theta \sigma_i^{(2)} \right) \dot{\theta} + \theta \sigma_{ij}^{(2)} \theta_{,j} \right],$$

provided $\Phi^{(1)} \neq 0$. From (3.16)–(3.19), (4.2)–(4.4), (4.8) and (4.9) it follows that

$$(4.10) \quad \kappa^{(0)} = 0, \quad \frac{\partial \sigma^{(0)}}{\partial \theta} = \frac{\sigma^{(1)}}{\Phi^{(1)}}, \quad \sigma_i^{(1)} = 0.$$

Further linearization can be carried out by assuming that the temperature change from the temperature T_0 of the natural state is small, i. e.

$$(4.11) \quad \theta = T_0 + T, \quad |T| \ll T_0, \quad T_0 > 0.$$

Then a linear theory with respect to e_{ij} and T requires

$$(4.12) \quad \begin{aligned} \sigma^{(0)} &= \sigma_0 - aT + \frac{1}{2} c_{ijkl} e_{ij} e_{kl} - a_{ij} e_{ij} T - \frac{1}{2} c \frac{T^2}{T_0}, \\ \sigma^{(1)} &= -b - h_{ij} e_{ij} - eT, \end{aligned}$$

$$(4.13) \quad \sigma^{(2)} = -f, \quad \sigma_i^{(2)} = -\frac{\alpha}{T_0} b_i, \quad \sigma_{ij}^{(2)} = \frac{\alpha}{T_0} k_{ij},$$

$$\Phi^{(1)} = \alpha + \beta T, \quad \Phi^{(2)} = \gamma,$$

$$(4.14) \quad \kappa_{ijkl}^{(1)} = d_{ijkl}, \quad \kappa_{ij}^{(1)} = -2b_{ij}, \quad \kappa_{ijk}^{(1)} = -2b_{ijk},$$

provided $\alpha \neq 0$ and assuming that the initial body is free from stress. The newly introduced coefficients are constants and

$$(4.15) \quad c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}, \quad a_{ij} = a_{ji}, \quad h_{ij} = h_{ji},$$

$$k_{ij} = k_{ji}, \quad d_{ijkl} = d_{jikl} = d_{ijlk} = d_{klij}, \quad b_{ij} = b_{ji}, \quad b_{ijk} = b_{jik},$$

where (4.5) and (4.6) are used.

With the help of (4.10)₂ we obtain

$$(4.16) \quad b = \alpha a, \quad h_{ij} = \alpha a_{ij}, \quad e = \frac{\alpha}{T_0} c + \beta a.$$

Then (4.2)–(4.4) take the form

$$(4.17) \quad \sigma = \sigma_0 - aT + \frac{1}{2} c_{ijkl} e_{ij} e_{kl} - a_{ij} e_{ij} T - \frac{1}{2} c \frac{T^2}{T_0} - \alpha a \dot{T} - \alpha a_{ij} e_{ij} \dot{T}$$

$$- e T \dot{T} - \frac{1}{2} f \dot{T}^2 - \frac{\alpha}{T_0} b_i \dot{T} T_{,i} + \frac{1}{2} \frac{\alpha}{T_0} k_{ij} T_{,i} T_{,j},$$

$$(4.18) \quad \Phi = T_0 + T + \alpha \dot{T} + \beta T \dot{T} + \frac{1}{2} \gamma \dot{T}^2,$$

$$(4.19) \quad t_{ij}^D = d_{ijkl} \dot{e}_{kl} - 2b_{ij} \dot{T} - 2b_{ijk} T_{,k}.$$

In order to have $\Phi > 0$ we add that $|\dot{T}| \ll T_0$.

From the equations (3.13), (3.14), (4.17), (4.19), (4.8), (4.9), (4.12) and (4.13) we obtain after linearization

$$(4.20) \quad t_{ij} = c_{ijkl} e_{kl} + d_{ijkl} \dot{e}_{kl} - a_{ij} T - (2b_{ij} + \alpha a_{ij}) \dot{T} - 2b_{ijk} T_{,k},$$

$$(4.21) \quad q_i = b_i \dot{T} - k_{ij} T_{,j},$$

$$(4.22) \quad \varrho_0 \eta = a + a_{ij} e_{ij} + c \frac{T}{T_0} + h \frac{\dot{T}}{T_0} + b_i \frac{T_{,i}}{T_0},$$

where

$$(4.23) \quad \frac{\alpha}{T_0} h = f - \gamma a.$$

With the help of (3.16), (4.17)–(4.19), (4.21) and (4.22) the inequality (3.15) yields

$$(4.24) \quad d_{ijkl} \dot{e}_{ij} \dot{e}_{kl} + \frac{\alpha c - h}{T_0} \dot{T}^2 + \frac{k_{ij}}{T_0} T_{,i} T_{,j} - 2b_{ij} \dot{e}_{ij} \dot{T} - 2b_{ijk} \dot{e}_{ij} T_{,k} - 2 \frac{b_i}{T_0} \dot{T} T_{,i} \geq 0.$$

Bearing in mind (4.17)–(4.19) and (4.20)–(4.22) we linearize the energy equation (3.22) to the form

$$(4.25) \quad h \ddot{T} - k_{ij} T_{,ij} + a_{ij} T_0 \dot{e}_{ij} + 2b_i \dot{T}_{,i} + c \dot{T} - \varrho_0 r = 0.$$

5. Linear Thermoviscoelasticity

The equations (4.1), (4.20) and (4.25) together with the equations of motion for the linear theory

$$(5.1) \quad t_{ij,j} + \rho_0 F_i = \rho_0 \ddot{u}_i$$

form a complete set of equations of the linear thermoviscoelasticity when a dependence on the temperature rate is taken into consideration.

If the initial body has a centre of symmetry at each point then

$$(5.2) \quad b_i = 0, \quad b_{ijk} = 0.$$

When the heat conduction equation (4.25) is not coupled with the strain field we have $a_{ij} = 0$. Since from (4.24) it follows that

$$(5.3) \quad k_{ij} T_{,i} T_{,j} \geq 0$$

we deduce that the heat conduction equation may be hyperbolic if

$$(5.4) \quad h > 0.$$

Then it will predict a finite speed of heat propagation.

For isotropic solids the constitutive coefficients are isotropic tensors so that we may write

$$(5.5) \quad \begin{aligned} c_{ijkl} &= \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \\ d_{ijkl} &= \lambda' \delta_{ij} \delta_{kl} + \mu' (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \\ a_{ij} &= \kappa \delta_{ij}, \\ b_{ij} &= \nu \delta_{ij}, \\ k_{ij} &= k \delta_{ij}. \end{aligned}$$

Then (4.20) and (4.25) reduce to

$$(5.6) \quad t_{ij} = (\lambda e_{kk} - \kappa T) \delta_{ij} + 2\mu e_{ij} + (\lambda' \dot{e}_{kk} - \kappa' \dot{T}) \delta_{ij} + 2\mu' \dot{e}_{ij},$$

$$(5.7) \quad h \ddot{T} - k T_{,ii} + \kappa T_0 \dot{e}_{ii} + c \dot{T} - \rho_0 r = 0,$$

where $\kappa' = 2\nu + \alpha\kappa$. The inequality (4.24) yields

$$(5.8) \quad \mu' \geq 0, \quad 3\lambda' + 2\mu' \geq 0, \quad \alpha c - h \geq 0, \quad (3\lambda' + 2\mu')(\alpha c - h) \geq 3\nu^2 T_0, \quad k \geq 0.$$

A uniqueness theorem can be proved in the linear case for the mixed boundary value problem under suitable restrictions on the coefficients.

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