

Electromechanical interaction in viscoelastic media

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Introduction

The intensive study of the electric properties of organic substances began about twenty years ago. A basis for this were the successes in the synthesis of new polymers, on the one hand, and the hope, on the other, that these studies might provide the key to the solution of a number of biologically important problems [1].

A great part of the organic substances manifest markedly expressed pyro- and piezo-electric properties [2—4]. A list of 400 new organic piezo electrics is given in the work [5]. Among the substances which manifest a piezo effect are substances of great biological importance: Desoxyribonucleic acid (DNA), Ribonucleic acid (RNA) and other nucleoproteins [6, 7]. The opinion exists that the piezo- and the pyro-effect play a fundamental role in biological functions [8].

A particularly strong piezo-effect has been discovered in organic polymers [9—13]. The piezo-electric effect is manifested both in the polar (polymethylmethacrylate and polyvinylchloride) and in the nonpolar polymers (polystyrene, polypropylene, polyethylene). As is known, a considerable part of these substances belong to the group of the viscoelastic bodies [14, 15]. Since the theory of the piezo-effect in viscoelastic bodies is not developed, in scientific publications this effect is examined on the basis of the electroelastic theory [16, 17] or with the aid of the theory of J. Kyame [18], which constitutes a generalization of the model of Voigt. The existing lacuna in the theory of constitutive equations and the thermodynamics of these media is the object of the present investigations of the authors.

Mathematical formalism

Some fundamental mathematical formulations, which are set forth in the works [19, 20], are used in the studies.

We introduce the space H :

$$\begin{aligned} & \langle b | \in H, | a \rangle \in H. \\ 1^0 \quad & | \lambda a \rangle + | \mu b \rangle = \mu | b \rangle + \lambda | a \rangle \in H; \\ & \langle \lambda a | + \langle \mu b | = \bar{\mu} \langle b | + \bar{\lambda} \langle a | \in H, \end{aligned}$$

$\lambda\mu \in C$ — the field of complex numbers.

$$\begin{aligned}
2^0 \quad & \exists |0\rangle, \quad \langle 0| \in H = \\
& 0|a\rangle = |0\rangle, \quad 0\langle a| = \langle 0| \\
& |a\rangle + |0\rangle = |a\rangle, \quad \langle a| + \langle 0| = \langle a|, \quad \forall |a\rangle, \quad \langle a| \in H; \\
3^0 \quad & (|a\rangle + |b\rangle) + |c\rangle = |a\rangle + (|b\rangle + |c\rangle) \\
& (\langle a| + \langle b|) + \langle c| = \langle a| + (\langle b| + \langle c|);
\end{aligned}$$

4° In $H = U_a H_a$ is introduced the bilinear form:

$$\varphi(\langle a| \in H_a; |b\rangle \in H_\beta) = \langle a|b\rangle \in C,$$

satisfying the condition

$$\begin{aligned}
\langle a|b\rangle &= 0 \quad \text{at } \alpha \neq \beta \\
&= > H = \oplus_\alpha H_\alpha.
\end{aligned}$$

$$5^0 \quad \text{In } H_a = H^0 \otimes G_a = H^0 \otimes G_a^1 \otimes G_a^2 \dots \otimes G_a^n,$$

there exists the basis

$$\begin{aligned}
|ti_1 i_2 \dots i_{n_a} \alpha\rangle &= |t\rangle |i_1 \alpha\rangle |i_2 \alpha\rangle \dots |i_{n_a} \alpha\rangle \\
|t\rangle &\in H^0, \quad |i_k \alpha\rangle \in G_a^k
\end{aligned}$$

with condition for norming

$$\begin{aligned}
\langle ai_{n_a} \dots i_1 t | t' j_1 \dots j_{n_a} \alpha \rangle &= \langle t | t' \rangle \langle ai_1 | j_1 \alpha \rangle \langle ai_2 | j_2 \alpha \rangle \dots \langle ai_{n_a} | j_{n_a} \alpha \rangle \\
&= \delta(t-t') \delta_{i_1 j_1} \delta_{i_2 j_2} \dots \delta_{i_{n_a} j_{n_a}}
\end{aligned}$$

and with condition for fulness

$$\exists \Psi_{i_1 i_2 \dots i_{n_a}}(t) \in 0^a = \rangle | \Psi \rangle = \Psi_{i_1 i_2 \dots i_{n_a}}(t) | ti_1 \dots i_{n_a} \rangle, \quad \forall | \Psi \rangle \in H_a.$$

0^a is the space of the generalized tensor functions with dimension n_a .

We will make use of Einstein rule for summation in the case of repeated indices (both discrete and indiscrete). In the case where the appropriate expression does not represent a sum under the repeating indices, we will put a line.

We introduce the following notations for the projection operators:

$$\begin{aligned}
P_a &= | \underline{a} \rangle \langle \underline{a} | = | i_1 \dots i_{n_a} \underline{a} \rangle \langle \underline{a} i_{n_a} \dots i_1 | : H \rightarrow H_a; \\
P_t &= | \underline{t} \rangle \langle \underline{t} | : H \rightarrow G; \\
P_{ta} &= | \underline{at} \rangle \langle \underline{ta} | : H \rightarrow G_a.
\end{aligned}$$

In accordance with the condition for fulness of the basis

$$(1) \quad \sum_a P_a = I, \quad \sum_t P_t = I, \quad \sum_{at} P_{at} = I,$$

where I is the identical operator.

6° In H^0 is defined the operator differentiation D

$$(2) \quad \langle t | D | t' \rangle = -\frac{\partial}{\partial t'} \delta(t-t'), \quad -\infty < t' < \infty.$$

Let us examine the problem of the own vectors of the operator D :

$$D | \omega \rangle = i\omega | \omega \rangle, \quad i\omega \in C.$$

Let us proceed to $|t\rangle$ representation with the aid of (1₂):

$$\langle t | D | t' \rangle \langle t' | \omega \rangle = i\omega \langle t | \omega \rangle.$$

We multiply the equation on the left by $|t\rangle$.

$$\langle t | D | t' \rangle \langle t' | \omega \rangle = i\omega \langle t | \omega \rangle$$

$$6^0 = \frac{\partial}{\partial t} \langle t | \omega \rangle = i\omega \langle t | \omega \rangle$$

$$= \lambda \langle t | \omega \rangle = \lambda e^{i\omega t}, \quad \lambda \in C.$$

We multiply the result obtained on the left by $|t\rangle$.

$$|t\rangle \langle t | \omega \rangle = \lambda e^{i\omega t} |t\rangle = | \omega \rangle = \lambda e^{i\omega t} |t\rangle$$

$$= | \omega \rangle \langle \omega | = \int_{-\infty}^{\infty} \lambda \bar{\lambda} e^{i(\omega t - \bar{\omega} t')} |t'\rangle \langle t| d\omega.$$

In the case when $\lambda = \frac{1}{\sqrt{2\pi}}$ and $\omega \in R$ — the set of the real numbers

$$\sum P_\omega = | \omega \rangle \langle \omega | = I.$$

Fundamental conceptions, principles and assumptions

The considerations to be made refer to the sufficiently small material volume ΔV belonging to the investigated body.

In accordance with the general theory of systems, the values which characterize a given system may be divided into three types: (a) external influences, (b) values which describe the reaction of the system, (c) values which describe the internal state of the system.

If the system ΔV is of a very small volume, it may be assumed that the system under consideration does not influence the self-congruent fields that are determined by the remaining part of the body and the external fields. In this case the mechanical stress, the electric field and the deviation of the temperature from its equilibrium value may be regarded as input influences, and the deformation, electric polarization and entropy as output reactions.

In the case of systems that possess only one degree of freedom, the thermodynamic state is determined unambiguously by the deformation tensor, polarization and entropy [21]. In this case the values describing the internal state coincide with those describing the output. In the general case viscoelastic bodies possess a large number of degrees of freedom. In the event of a finite number of degrees of freedom, by state of the system we will understand the totality of the states of the individual subsystems, each of them containing just one degree of freedom. This idea is set forth in the concluding notes of this paper [22].

We will accept the validity of the following principles and assumptions:

Principles

- I. Principles of causality.
- II. First principle of thermodynamics.
- III. Second principle of thermodynamics.

Assumptions for the system.

- 1° The system is linear and smooth.
- 2° The system is invariant with respect to the operation translation in time.
- 3° The system is made up of a finite number of subsystems each of which has not internal degree of freedom.
- 4° Since mechanical stress, temperature and electric field are intensive quantities, all subsystems and the aggregate system are characterized by general values of the input influences.
- 5° The reaction of the aggregate system is a sum of the reactions of the individual subsystems.
- 6° The physical quantities which independently of each other describe the state of input and output are represented with the aid of orthogonal subspaces of H .

Notations:

$$|\zeta\rangle = \sigma_{ij}(t) |tij\varepsilon\rangle + \mathcal{E}_i(t) |tip\rangle + \Delta T(t) |ts\rangle \in H = H_\varepsilon \oplus H_p \oplus H_{ts},$$

- $|\zeta\rangle$ — vector of input influence;
- $\sigma_{ij}(t)$ — stress tensor;
- $\mathcal{E}_i(t)$ — electric field power;
- $\Delta T(t)$ — deviation of temperature from its equilibrium value.

$$|\xi\rangle = \varepsilon_{ij}(t) |tij\rangle + P_i(t) |tip\rangle + S(t) |ts\rangle \in H,$$

- $|\xi\rangle$ — vector of output reaction or output state;
- $\varepsilon_{ij}(t)$ — deformation tensor;
- $P_i(t)$ — electric polarization vector;
- $S(t)$ — entropy density.

$$|\xi^k\rangle = \varepsilon_{ij}^k(t) |tij\varepsilon\rangle + P_i^k(t) |tip\rangle + S^k(t) |ts\rangle —$$

vector of the output reaction or output state for the k th subsystem.

$$|X\rangle = (|\xi^1\rangle, |\xi^2\rangle, \dots, |\xi^N\rangle) \in X;$$

- $|X\rangle$ — state of the system;
- N — number of subsystems;
- X — space of the states.

Constitutive equations

According to the general system theory, every linear, smooth, finite-dimensional dynamic system, may be described in a temporary presentation by the equations [23]:

$$(4) \quad \frac{d}{dt} |X(t)\rangle = F |X(t)\rangle + G |\zeta(t)\rangle$$

$$|\xi(t)\rangle = \Gamma |X(t)\rangle,$$

where F , G , Γ , in the case of systems which are invariant with respect to the operation translation in time, are real operators independent of time.

In an operator form (4) is presented by

$$(5) \quad \begin{aligned} D|X\rangle &= F|X\rangle + G|\xi\rangle \\ |\xi\rangle &= I|X\rangle. \end{aligned}$$

An analogous equation may be written for the k th subsystem:

$$\begin{aligned} D|X^k\rangle &= F^k|X^k\rangle + G^k|\xi^k\rangle \\ |\xi^k\rangle &= I^k|X^k\rangle. \end{aligned}$$

According to the assumptions made

$$I^k = 1, \quad |\xi^k\rangle = |\xi\rangle.$$

We have therefore

$$(6) \quad \begin{aligned} D|X^k\rangle &= F^k|X^k\rangle + G^k|\xi\rangle; \\ |\xi^k\rangle &= |X^k\rangle. \end{aligned}$$

It follows from assumption (5⁰) and from (6) that the system (5) is of the type:

$$(7) \quad \begin{aligned} D(|\xi^1\rangle, |\xi^2\rangle, \dots, |\xi^N\rangle) &= \begin{pmatrix} F^1 & & 0 \\ & F^2 & \\ & & \ddots \\ 0 & & & F^N \end{pmatrix} \begin{pmatrix} |\xi^1\rangle \\ |\xi^2\rangle \\ \vdots \\ |\xi^N\rangle \end{pmatrix} + (G^1, G^2, \dots, G^N)|\xi\rangle \\ |\xi\rangle &= (1, 1, 1, 1, \dots, 1) \begin{pmatrix} |\xi^1\rangle \\ |\xi^2\rangle \\ \vdots \\ |\xi^N\rangle \end{pmatrix} \end{aligned}$$

In ω presentation from (6₁) and (6₂) we obtain

$$(8) \quad (i\omega - F^k)|\xi^k(\omega)\rangle = G^k|\xi(\omega)\rangle = |\xi^k(\omega)\rangle = R_{F^k}(\omega)G^k|\xi(\omega)\rangle,$$

where we assume that the resolvent R_{F^k} of the operator F^k is defined.

After substituting (8) in (7₂) we obtain:

$$(9) \quad |\xi(\omega)\rangle = R_{F^k}G^k|\xi(\omega)\rangle.$$

Differential form of constitutive equations

Let us multiply the left and the right side of (9) by the product

$$\prod_{k=1}^N (i\omega\eta^k + E^k),$$

where η^k is the inverse operator of G^k , and $E^k = -(G^k)^{-1}F^k$ (it is assumed that G^k possesses an inverse operator). If moreover the commutation relations have been fulfilled

$$[(i\omega\eta^k + E^k), (i\omega\eta^l + E^l)] = 0,$$

we obtain

$$\prod_{k=1}^N (i\omega\eta^k + E^k) |\xi(\omega)\rangle = \sum_{k=1}^N \prod_{l \neq k}^N (i\omega\eta^l + E^l) |\xi(\omega)\rangle.$$

After developing the products we obtain

$$(10) \quad \sum_{k=0}^N A^k (i\omega)^k |\xi(\omega)\rangle = \sum_{k=0}^{N-1} B^k (i\omega)^k |\xi(\omega)\rangle,$$

where the operators A^k and B^k are products and sums of the operators η^k and E^k . We multiply both sides of (10) by $|\omega\rangle$:

$$\begin{aligned} \sum_{k=0}^N A^k (i\omega)^k |\xi(\omega)\rangle |\omega\rangle &= \sum_{k=0}^{N-1} B^k (i\omega)^k |\xi(\omega)\rangle |\omega\rangle. \\ \sum_{k=0}^N A^k (i\omega)^k |\xi(\omega)\rangle |\omega\rangle &= \sum_{k=0}^N A^k (i\omega)^k |\omega\rangle |\xi(\omega)\rangle = \sum_{k=0}^N A^k D^k |\omega\rangle |\xi(\omega)\rangle = \sum_{k=0}^N A^k D^k |\xi\rangle. \end{aligned}$$

Analogously

$$\sum_{k=0}^{N-1} B^k (i\omega)^k |\xi(\omega)\rangle |\omega\rangle = \sum_{k=0}^{N-1} B^k D^k |\xi\rangle.$$

The constitutive equation, consequently, may be written in the form:

$$(11) \quad \sum_{k=0}^N A^k D^k |\xi\rangle = \sum_{k=0}^{N-1} B^k D^k |\xi\rangle.$$

Let us proceed to $|t\rangle$ presentation

$$\begin{aligned} \sum_{k=0}^N A^k \langle t | D^k | t' \rangle \langle t' | \xi \rangle &= \sum_{k=0}^{N-1} B^k \langle t | D^k | t' \rangle \langle t' | \xi \rangle. \\ (12) \quad \langle t | D^k | t' \rangle &= \langle t | D | t^1 \rangle \langle t^1 | D | t^2 \rangle \dots \langle t^{k-1} | D | t' \rangle \\ &= (-1)^k \frac{\partial}{\partial t^1} \delta(t - t^1) \frac{\partial}{\partial t^2} \delta(t^1 - t^2) \dots \frac{\partial}{\partial t^k} \delta(t^{k-1} - t') = (-1)^k \frac{\partial}{\partial t^k} \delta(t - t'). \end{aligned}$$

After substituting this result in (12) we obtain:

$$(13) \quad \sum_{k=0}^N A^k \frac{d^k}{dt^k} |\xi(t)\rangle = \sum_{k=0}^{N-1} B^k \frac{d^k}{dt^k} |\xi(t)\rangle$$

$$\begin{aligned} &= \sum_{k=0}^N A^k \frac{d^k}{dt^k} (\epsilon_{k'l}(t) |k'l\rangle + P_{k'}(t) |k'p\rangle + S(t) |s\rangle) \\ (14) \quad &= \sum_{k=0}^{N-1} B^k \frac{d^k}{dt^k} (\sigma_{k'l}(t) |k'l\rangle + \mathcal{E}_{k'}(t) |k'p\rangle + AT(t) |s\rangle). \end{aligned}$$

After projecting (14) in succession in G_s , G_p and G_s we obtain the set of differential equations:

$$\begin{aligned}
& \sum_{k=0}^N A_{ijrs}^{\varepsilon \varepsilon k} \frac{d^k}{dt^k} \varepsilon_{rs} + \sum_{k=0}^N A_{ijr}^{\varepsilon p k} \frac{d^k}{dt^k} P_r + \sum_{k=0}^N A_{ij}^{\varepsilon s k} \frac{d^k}{dt^k} S \\
&= \sum_{k=0}^{N-1} B_{ijrs}^{\varepsilon \varepsilon k} \frac{d^k}{dt^k} \sigma_{rs} + \sum_{k=0}^{N-1} B_{ijr}^{\varepsilon p k} \frac{d^k}{dt^k} \mathcal{E}_r + \sum_{k=0}^{N-1} B_{ij}^{\varepsilon s k} \frac{d^k}{dt^k} \Delta T; \\
(15) \quad & \sum_{k=0}^N A_{irs}^{p \varepsilon k} \frac{d^k}{dt^k} \varepsilon_{rs} + \sum_{k=0}^N A_{ir}^{p p k} \frac{d^k}{dt^k} P_r + \sum_{k=0}^N A_i^{p s k} \frac{d^k}{dt^k} S \\
&= \sum_{k=0}^{N-1} B_{irs}^{p \varepsilon k} \frac{d^k}{dt^k} \sigma_{rs} + \sum_{k=0}^{N-1} B_{ir}^{p p k} \frac{d^k}{dt^k} \mathcal{E}_r + \sum_{k=0}^{N-1} B_i^{p s k} \frac{d^k}{dt^k} \Delta T; \\
& \sum_{k=0}^N A_{rs}^{s \varepsilon k} \frac{d^k}{dt^k} \varepsilon_{rs} + \sum_{k=0}^N A_r^{s p k} \frac{d^k}{dt^k} P_r + \sum_{k=0}^N A^{s s k} \frac{d^k}{dt^k} S \\
&= \sum_{k=0}^{N-1} B_{rs}^{s \varepsilon k} \frac{d^k}{dt^k} \sigma_{rs} + \sum_{k=0}^{N-1} B_r^{s p k} \frac{d^k}{dt^k} \mathcal{E}_r + \sum_{k=0}^{N-1} B^{s s k} \frac{d^k}{dt^k} \Delta T.
\end{aligned}$$

In the special case where the input of the system is represented only by the mechanical stress, and the output only by the deformation, we shall demonstrate that from (10) may be obtained the equations of the well-known rheological models. We take $N=3$. We repeat the procedure and obtain:

$$\sum_{k=0}^3 A_{ijrs}^k \frac{d^k}{dt^k} \varepsilon_{rs} = \sum_{k=0}^2 B_{ijrs}^k \frac{d^k}{dt^k} \sigma_{rs},$$

where

$$\begin{aligned}
A_{ijkl}^3 &= \langle ij | \eta^1 \eta^2 \eta^3 | kl \rangle; \\
A_{ijkl}^2 &= \langle ij | \eta^1 \eta^2 E^3 + E^1 \eta^2 \eta^3 + \eta^1 E^2 \eta^3 | kl \rangle; \\
(16) \quad A_{ijkl}^1 &= \langle ij | \eta^1 E^2 E^3 + E^1 \eta^2 E^3 + E^1 E^2 \eta^3 | kl \rangle; \\
A_{ijkl}^0 &= \langle ij | E^1 E^2 E^3 | kl \rangle; \\
B_{ijkl}^2 &= \langle ij | \eta^1 \eta^2 + \eta^2 \eta^3 + \eta^1 \eta^3 | kl \rangle; \\
B_{ijkl}^1 &= \langle ij | \eta^1 E^2 + E^1 \eta^2 + \eta^1 E^3 + \eta^2 E^3 + E^2 \eta^3 + E^1 \eta^3 | kl \rangle; \\
B_{ijkl}^0 &= \langle ij | E^1 E^2 + E^1 E^3 + E^2 E^3 | kl \rangle.
\end{aligned}$$

In the case of $\|\eta^1\| \rightarrow 0^+$ we obtain the five-parameter model of G. Brankov.

$$\sum_{k=0}^2 A_{ijrs}^k \frac{d^k}{dt^k} \varepsilon_{rs} = \sum_{k=0}^2 B_{ijrs}^k \frac{d^k}{dt^k} \sigma_{rs},$$

with coefficients

$$\begin{aligned}
A_{ijkl}^2 &= \langle ij | E^1 \eta^2 \eta^3 | kl \rangle; \\
A_{ijkl}^1 &= \langle ij | E^1 \eta^2 E^3 + E^1 E^2 \eta^3 | kl \rangle;
\end{aligned}$$

$$\begin{aligned}
(17) \quad A_{ijkl}^0 &= \langle ij | E^1 E^2 E^3 | kl \rangle; \\
B_{ijkl}^2 &= \langle ij | \eta^2 \eta^3 | kl \rangle; \\
B_{ijkl}^1 &= \langle ij | E^1 \eta^1 + E^2 \eta^3 + \eta^2 E^3 + E^2 \eta^3 | kl \rangle; \\
B_{ijkl}^0 &= \langle ij | E^1 E^2 + E^1 E^3 + E^2 E^3 | kl \rangle.
\end{aligned}$$

The models of Zener, Burgers, Maxwell and Voigt are obtained in an analogous manner.

Integral form of the constitutive equations

From (8) we have

$$|\xi^k\rangle = R_{F^k}(\omega) G^k |\zeta(\omega)\rangle |\omega\rangle.$$

Let us switch over to $|t\rangle$ presentation

$$\begin{aligned}
\langle t | \xi^k \rangle &= R_{F^k}(\omega) G^k \langle t | \omega \rangle \langle \omega | t' \rangle \langle t' | \zeta \rangle; \\
|\xi^k(t)\rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{F^k}(\omega) G^k e^{i\omega(t-t')} |\zeta(t')\rangle dt' d\omega.
\end{aligned}$$

The equation may be written in the compact form

$$(18) \quad |\xi^k(t)\rangle = \int_{-\infty}^{\infty} U^k(t-t') |\zeta(t')\rangle dt',$$

where

$$U^k(t-t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{F^k}(\omega) G^k e^{i\omega(t-t')} d\omega.$$

From (18) we see that the moment state of the output depends on the input state in the time interval $t' \in (-\infty, \infty)$. According to the principle of causality, however, the output state is uniquely determined by the input state in the time interval $t' \in (-\infty, t]$.

Let us examine two possible trajectories of the input $|\zeta\rangle$ and $|\zeta'\rangle$, which meet the condition

$$\begin{aligned}
\langle t' | \zeta \rangle &= \langle t' | \zeta' \rangle \quad \text{at } t' \leq t \\
\langle t' | \zeta \rangle &\neq \langle t' | \zeta' \rangle \quad \text{at } t' > t.
\end{aligned}$$

From the principle of causality and from (18) follows

$$(19) \quad \int_t^{\infty} U^k(t-t') (|\zeta'(t')\rangle - |\zeta(t')\rangle) dt' = 0.$$

Since $|\zeta'(t)\rangle$ and $|\zeta(t)\rangle$ are in general different at $t' \in (t, \infty)$ from the equation we have:

$$U^k(t-t') = 0 \quad \text{for } t' > t.$$

As is to be seen, this leads to constraints with respect to the operators F^k which take part in (7).

At $t-t' < 0$

$$U^k(t-t') = -\lim_{R \rightarrow \infty} \frac{1}{2\pi i} \oint_{ABCA} R_{F^k}(z) G^k e^{z(t-t')} dz.$$

ABC is a semicircumference with $\text{Re} z \geq 0$

According to the generalized theorem of Cauchy in case where F^k are normal operators

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi i} \oint_{ABCA} R_{F^k}(z) e^{z(t-t')} dz = e^{(t-t') \sum_m \lambda_m^k \hat{P}_m^k},$$

where by λ_m^k we have denoted the points of the spectrum of F^k lying in the complex semiplane $x > 0$, and by $\hat{P}_m^k$ — the projection operators with the aid of which the expansion of F^k is realized. From the last equation it is seen that the necessary and adequate condition for the fulfilment of the principle of causality is that there should exist no points of the spectrum of F^k lying in the complex semiplane $x > 0$.

At $t-t' > 0$:

$$(20) \quad U^k(t-t') = \lim_{R \rightarrow \infty} \frac{1}{2\pi i} \oint_{ACDA} R_{F^k}(z) G^k e^{z(t-t')} dz \\ = e^{\lambda_m^k (t-t')} \hat{P}_m^k G^k = K^{km} e^{-\frac{(t-t')}{\tau_{km}^k}},$$

where ADC is a semicircumference with $\text{Re} z \leq 0$

λ_m^k — the points of the spectrum of F^k lying in the semiplane $x < 0$.

$\hat{P}_m^k$ — their corresponding projection operators

$$K^{km} = \hat{P}_m^k G^k; \quad \tau_{km}^k = -\frac{1}{\lambda_m^k}.$$

We will call the values τ_{km}^k relaxation times.

In the case where the state of the system is physically observable, the operator $U^k(t-t')$ in the temporary presentation is a real operator. It follows from this that the points λ_m^k of the spectrum of F^k are real. On the other hand all eigenvalues of F^k lie in the semiplane $x < 0$. This means that all eigenvalues λ_m^k are arranged on the axis $x < 0$. It follows from this that the memory of the system fades with time. This result is used as a fundamental principle in the works of B. Coleman and Noll [24, 25, 26].

At $t = t'$

$$U^k(0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} R_{F^k}(\omega) G^k d\omega = \lim_{R \rightarrow \infty} \frac{1}{2\pi i} \oint_{ACDA} R_{F^k}(z) G^k dz.$$

We will demonstrate that $\|U^k(0)\|$ is limited if $\|G^k\|$ is limited.

$$\|U^k(0)\| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \|R_{F^k}(\omega)\| \|G^k\| d\omega \leq \lim_{R \rightarrow \infty} \frac{1}{2\pi} \oint_{ACDA} \|R_{F^k}(z)\| \|G^k\| |dz|.$$

On the other hand, as there are no points of the spectrum of the operator F^k in the semiplane $x > 0$, the equation is fulfilled

$$\lim_{R \rightarrow \infty} \oint_{\text{ACDA}} R_{F^k}(z) dz = \lim_{R \rightarrow \infty} \oint_{\text{ABCD A}} R_{F^k}(z) dz.$$

As is known from the functional analysis [19] if the norm of the operator T is limited by α ,

$$\|T\| \leq \alpha < 1$$

the reverse transformation of $(I - T)$ is limited and satisfies the condition

$$\|(I - T)^{-1}\| \leq \frac{1}{1 - \alpha}.$$

Let us examine $\|R_{F^k}\|$ on the circumference ABCDA

$$\|R_{F^k}\| = \|(I_z - F^k)^{-1}\| = \frac{1}{|z|} \left\| \left(I - \frac{F^k}{z} \right)^{-1} \right\|.$$

At

$$|z| \rightarrow \infty \quad \left\| \frac{F}{2z} \right\| \rightarrow 0.$$

Consequently

$$\lim_{R \rightarrow \infty} \left\| \left(I - \frac{F}{z} \right)^{-1} \right\| = 1 = >$$

$$\lim_{R \rightarrow \infty} \oint_{\text{ABCD A}} \|R_{F^k}(z)\| |dz| = \oint_{\text{ABCD A}} \frac{|dz|}{|z|} = 2\pi.$$

From the last equation follows the inequality

$$\|U(0)\| \leq \lim_{R \rightarrow \infty} \frac{1}{2\pi} \oint_{\text{ABCD A}} \|R_{F^k}(z)\| \|G^k\| |dz| = \|G^k\|.$$

Therefore in the semiclosed interval $(-\infty, t]$ for $U^k(t - t')$ we obtain :

$$U^k(t - t') = K^{\frac{k}{m}} e^{-\frac{(t-t')}{\tau \frac{k}{m}}} + \theta^k(t - t'),$$

where the operator θ has only one value different from zero at $t = t'$ in which its norm is limited.

From (18) and (19) we have

$$\begin{aligned} |\xi^k(t)\rangle &= \int_{-\infty}^t \left(K^{\frac{k}{m}} e^{-\frac{(t-t')}{\tau \frac{k}{m}}} + \theta^k(t - t') \right) |\zeta(t')\rangle dt' \\ (21) \quad &= \lim_{\Delta \rightarrow 0} \int_{-\infty}^{t-\varepsilon} K^{\frac{k}{m}} e^{-\frac{(t-t')}{\tau \frac{k}{m}}} |\zeta(t')\rangle dt' + \lim_{\varepsilon \rightarrow 0} \int_{t-\varepsilon}^t \theta |\zeta(t')\rangle dt' \\ &= \int_{-\infty}^t K^{\frac{k}{m}} e^{-\frac{(t-t')}{\tau \frac{k}{m}}} |\zeta(t')\rangle dt'. \end{aligned}$$

If the operator $\|G^k\| \rightarrow \infty$ the integral (18) is divergent (at $\langle \zeta(t') | \zeta(t') \rangle \neq 0$ in the uninterrupted interval $(a, b] \in (-\infty, t]$ its value is regulated if $\tau^{km} \rightarrow 0+$. By this a delta-shaped singularity is produced:

$$\begin{aligned} U^k(t-t') &= A^k \delta(t-t') \\ \Rightarrow |\xi^k(t)\rangle &= A^k |\zeta^k(t)\rangle. \\ |\xi(t)\rangle &= \sum_{k=1}^N |\xi^k(t)\rangle = \sum_{k=1}^{N_1} A^k |\zeta(t)\rangle + \sum_{k=1}^{N-N_1} \int_{-\infty}^t K^{km} e^{-\frac{(t-t')}{\tau^{km}}} |\zeta(t')\rangle dt'. \end{aligned}$$

We substitute $|\xi(t)\rangle$ and $|\zeta(t)\rangle$ by their expansion and project the equation consecutively in G_ε , G_p and G_s . The following is obtained:

$$\begin{aligned} \varepsilon_{ij}(t) &= A_{ijkl}^e \sigma_{kl}(t) + A_{ijr}^e \mathcal{E}_r(t) + A_{ij}^e \Delta T(t) \\ &+ \int_{-\infty}^t e^{-\frac{(t-t')}{\tau^{km}}} (B_{ijrs}^{km} \sigma_{rs}(t') + B_{ijr}^{km} \mathcal{E}_r(t') + B_{ij}^{km} \Delta T(t')) dt'; \\ P_i(t) &= A_{ijr}^p \sigma_{jr}(t) + A_{ij}^p \mathcal{E}_j(t) + A_i^p \Delta T(t) \\ (22) \quad &+ \int_{-\infty}^t e^{-\frac{(t-t')}{\tau^{km}}} (F_{ijr}^{km} \sigma_{jr}(t') + F_{ij}^{km} \mathcal{E}_j(t') + F_i^{km} \Delta T(t')) dt'; \\ S(t) &= A_{ij}^s \sigma_{ij}(t) + A_i^s \mathcal{E}_i(t) + A^s \Delta T(t) \\ &+ \int_{-\infty}^t e^{-\frac{(t-t')}{\tau^{km}}} (L_{ij}^{km} \sigma_{ij}(t') + L_i^{km} \mathcal{E}_i(t') + L^{km} \Delta T(t')) dt'. \end{aligned}$$

In the examinations that follow we will make use of the compact notation

$$(23) \quad |\xi(t)\rangle = \int_{-\infty}^t K^{km} e^{-\frac{(t-t')}{\tau^{km}}} |\zeta(t')\rangle dt',$$

where the sign] symbolically expresses the assumption that a singular part may exist to the integral.

Fundamental Thermodynamic Relations

It follows from (6) that the input influence in each subsystem may be presented as the sum of two members:

$$|\zeta\rangle = |\zeta^{k(r)}\rangle + |\zeta^{k(ir)}\rangle,$$

the first of which

$$(24) \quad |\zeta^{k(r)}\rangle = -(G^k)^{-1} F^k |\xi^k\rangle = E^k |\xi^k\rangle$$

does not depend on the operator differentiation, and the second

$$(25) \quad |\zeta^{k(ir)}\rangle = D(G^k)^{-1} |\xi^k\rangle = D\eta^k |\xi^k\rangle$$

is a homogeneous linear function of D .

If we project (24) and (25) in $\hat{G}_e$, $\hat{G}_p$ and $\hat{G}_s$ we obtain the following set of differential equations:

$$\begin{aligned}
 \sigma_{ij}^{k(r)}(t) &= E_{ijrs}^{k\varepsilon} \varepsilon_{rs}^k(t) + E_{ijr}^{kr} P_r^k(t) + E_{ij}^{k\varepsilon} S^k(t); \\
 \mathcal{E}_i^{k(r)}(t) &= E_{ijr}^{kp} \varepsilon_{jr}^k(t) + E_{ij}^{kp} P_j^k(t) + E_i^{kp} S^k(t); \\
 \Delta T^{k(r)} &= E_{ij}^{ks} \varepsilon_{ij}^k(t) + E_i^{ks} P_i^k(t) + E^{ks} S^k(t); \\
 \sigma_{ij}^{k(ir)}(t) &= \eta_{ijrs}^{k\varepsilon} \frac{d}{dt} \varepsilon_{rs}^k(t) + \eta_{ijr}^{kr} \frac{d}{dt} P_r^k(t) + \eta_{ij}^{k\varepsilon} \frac{d}{dt} S^k; \\
 \mathcal{E}_i^{k(ir)}(t) &= \eta_{ijr}^{kp} \frac{d}{dt} \varepsilon_{jr}^k(t) + \eta_{ij}^{kp} \frac{d}{dt} P_j^k(t) + \eta_i^{kp} \frac{d}{dt} S^k; \\
 \Delta T^{k(ir)} &= \eta_{ij}^{ks} \frac{d}{dt} \varepsilon_{ij}^k(t) + \eta_i^{ks} \frac{d}{dt} P_i^k + \eta^{ks} \frac{d}{dt} S^k,
 \end{aligned}
 \tag{25'}$$

where

$$\begin{aligned}
 \varepsilon_{ij}^k(t) &= \int_{-\infty}^t e^{-\frac{(t-t')}{\tau^{km}}} (B_{ijrs}^{km} \sigma_{rs}(t') + B_{ijr}^{km} \mathcal{E}_r(t') + B_{ij}^{km} \Delta T(t')) dt'; \\
 P_i^k(t) &= \int_{-\infty}^t e^{-\frac{(t-t')}{\tau^{km}}} (F_{ijr}^{km} \sigma_j(t') + F_{ij}^{km} \mathcal{E}_i(t') + F_i^{km} \Delta T(t')) dt'; \\
 S^k(t) &= \int_{-\infty}^t e^{-\frac{(t-t')}{\tau^{km}}} (L_{ij}^{km} \sigma_{ij}(t') + L_i^{km} \mathcal{E}_i(t') + L^{km} \Delta T(t')) dt'.
 \end{aligned}
 \tag{26}$$

According to the first and the second principle of thermodynamics, the following relation exists among the internal energy, the work of the forces which are not dependent on velocity, temperature and entropy:

$$\begin{aligned}
 dH &= \sum_{k=1}^N (\sigma_{ij}^{k(r)} d\varepsilon_{ij}^k + \mathcal{E}_i^{k(r)} dP_i^k) + TdS, \\
 H &= H(\varepsilon_{ij}^k, P_i^k, S^k).
 \end{aligned}
 \tag{27}$$

It follows from (27) that T does not depend on the operator differentiation and constitutes a reversible thermodynamic force. In (25') we consequently have:

$$\begin{aligned}
 \eta_{ij}^{ks} &= 0, \quad \eta_i^{ks} = 0, \quad \eta^{ks} = 0, \\
 \Delta T(t) &= \Delta T^{k(r)}(t) = E_{ij}^{ks} \varepsilon_{ij}^k(t) + E_i^{ks} P_i^k(t) + E^{ks} S^k(t) \\
 &= \frac{1}{E^{ks}} (\Delta T(t) - E_{ij}^{ks} \varepsilon_{ij}^k(t) - E_i^{ks} P_i^k(t)).
 \end{aligned}
 \tag{28}$$

After summing by k we obtain:

$$S(t) = \left(\sum_k \frac{1}{E^{ks}} \right) \Delta T(t) - \sum_k \frac{1}{E^{ks}} E_{ij}^{ks} \varepsilon_{ij}^k(t) - \sum_k \frac{1}{E^{ks}} E_i^{ks} P_i^k(t).
 \tag{29}$$

According to the principles of thermodynamics

$$H = \sum_{k=1}^N H^k$$

where by H^k we have denoted the internal energy of the k th subsystem. In quadratic approximation

$$(30) \quad H^k = \frac{1}{2} H_{ijrs}^k \varepsilon_{ij}^k \varepsilon_{rs}^k + \frac{1}{2} H_{ij}^k P_i^k P_j^k + \frac{1}{2} H^k S^k S^k + V_{ijr}^k \varepsilon_{ij}^k P_r^k + V_{ij}^k \varepsilon_{ij}^k S^k + V_i^k P_i^k S^k + \Pi_{ij}^k \varepsilon_{ij}^k + \Pi_i^k P_i^k + \Pi^k S^k$$

$$H_{ijrs}^k = H_{rsij}^k, \quad H_{ij}^k = H_{ji}^k.$$

From (27) and (30) we have

$$(31) \quad \sigma_{ij}^{k(r)} = H_{ijrs}^k \varepsilon_{rs}^k + V_{ijr}^k P_r^k + V_{ij}^k S^k + \Pi_{ij}^k;$$

$$\sigma_i^{k(r)} = H_{ij}^k P_j^k + V_{rsi}^k \varepsilon_{rs}^k + V_i^k S^k + \Pi_i^{kp};$$

$$T = H^k S^k + V_{ij}^k \varepsilon_{ij}^k + V_i^k P_i^k + \Pi^k.$$

From the comparison of these three equations with (25) we obtain:

$$(32) \quad H_{ijrs}^k = E_{ijrs}^{ke}, \quad H_{ij}^k = E_{ij}^{kp}, \quad H^k = E^{ks};$$

$$V_{ijr}^k = E_{rij}^{ke}, \quad V_{ij}^k = E_{ij}^{ke}, \quad V_i^k = E_i^{kp};$$

$$\Pi_{ij}^k = 0, \quad \Pi_i^k = 0, \quad \Pi^k = T_0;$$

$$E_{ij}^{ks} = E_{ij}^{ks}, \quad E_{ijr}^{ke} = E_{rij}^{kp}, \quad E_i^{kp} = E_i^{ks};$$

$$E_{ijrs}^{ke} = E_{rsij}^{ke}, \quad E_{ij}^{kp} = E_{ji}^{kp}$$

$$\Rightarrow H^k = \langle \xi^k(t) | E^k | \xi^k(t) \rangle + T_0 S^k(t),$$

T_0 is the equilibrium temperature.

After making use of (18) and (20) we obtain

$$(33) \quad H^k = \int_{-\infty}^t \int_{-\infty}^t \langle \zeta(t') | K^{+k m} E^k K^{k n} | \zeta(t'') \rangle e^{-\frac{t-t'}{\tau^{k m}}} e^{-\frac{t-t''}{\tau^{k n}}} dt' dt''$$

$$+ T_0 \int_{-\infty}^t \langle s | K^{k m} | \zeta(t') \rangle e^{-\frac{t-t'}{\tau^{k m}}} dt'.$$

The summation of (33) by index k gives us the full internal energy.

From (33) we may obtain an expression for the free energy. By definition the free energy $\mathcal{F}$ is equal to

$$\mathcal{F} = H - TS.$$

After substituting H and S by their equals we obtain

$$\mathcal{F} = \langle \xi^k(t) | E^k | \xi^k(t) \rangle + T_0 S - TS = \langle \xi^k(t) | E^k | \xi^k(t) \rangle - \Delta TS$$

$$(34) \quad = \int_{-\infty}^t \int_{-\infty}^{t'} \langle \zeta(t') | K^{+km} E^k K^{kn} | \zeta(t'') \rangle e^{-\frac{t-t'}{\tau^{km}}} e^{-\frac{t-t''}{\tau^{kn}}} dt' dt'' \\ - \int_{-\infty}^t \langle \zeta(t) | s \rangle \langle s | K^{km} | \zeta(t') \rangle e^{-\frac{t-t'}{\tau^{km}}} dt'.$$

The remaining thermodynamic potentials may be obtained in an analogous manner.

(by K^+ is indicated the adjoint operator of K).

In the thermodynamics of irreversible processes a particularly important role is played by the dissipative function, introduced by Rayleigh, with the aid of which the irreversible forces are expressed as gradients in the space of the velocities.

$$(35) \quad D \left(\frac{d}{dt} \varepsilon_{ij}^k(t); \frac{d}{dt} P_j^k(t) \right) = \sigma_{ij}^{k(ir)} \frac{d}{dt} \varepsilon_{ij}^k + \zeta_i^{k(ir)} \frac{dP_i^k}{dt}.$$

After using (25') we obtain

$$D = \eta_{ijrs}^{ke} \frac{d}{dt} \varepsilon_{ij}^k \frac{d}{dt} \varepsilon_{rs}^k + \eta_{ijr}^{ke} \frac{d}{dt} \varepsilon_{ij}^k \frac{d}{dt} P_r + \eta_{ij}^{ke} \frac{d}{dt} \varepsilon_{ij}^k \frac{d}{dt} S^k \\ + \eta_{ijr}^{kp} \frac{d}{dt} P_r \frac{d}{dt} \varepsilon_{ij}^k + \eta_{ij}^{kp} \frac{d}{dt} P_i^k \frac{d}{dt} P_j^k + \eta_i^{kp} \frac{dS^k}{dt} \frac{dP_i^k}{dt}.$$

The velocities of entropy change take part in the right part of this equation. On the other hand D does not depend on them. Consequently

$$(36) \quad \eta_{ij}^{ke} = 0; \quad \eta_i^{kp} = 0.$$

As is demonstrated in [22] the irreversible forces may be presented as gradients in the space of the velocities of a scalar function depending on the velocities only through D . We, therefore, have

$$(37) \quad \sigma_{ij}^{k(ir)} = \frac{\partial \varphi}{\partial D} \frac{dD}{d \left(\frac{d\varepsilon_{ij}^k}{dt} \right)}, \quad \zeta_i^{k(ir)} = \frac{\partial \varphi}{\partial D} \frac{dD}{d \left(\frac{dP_i^k}{dt} \right)} = \left\langle \frac{\partial \sigma_{ij}^{k(ir)}}{\partial \left(\frac{dP_r^k}{dt} \right)} = \frac{\partial \zeta_r^{k(ir)}}{\partial \left(\frac{d\varepsilon_{ij}^k}{dt} \right)} = \eta_{ijr}^{ke} = \eta_{rij}^{kp} = \eta_{ijr}^k \right\rangle.$$

After making use of this result in the expression for D we obtain

$$D = \eta_{ijrs}^{ke} \frac{d\varepsilon_{ij}^k}{dt} \frac{d\varepsilon_{rs}^k}{dt} + \eta_{ijr}^{ke} \frac{dP_i^k}{dt} \frac{dP_j^k}{dt} + 2\eta_{ijr}^k \frac{d\varepsilon_{ij}^k}{dt} \frac{dP_r^k}{dt} = \langle \varepsilon_{ij} | \eta^k | r s \varepsilon \rangle \frac{d\varepsilon_{ij}^k}{dt} \frac{d\varepsilon_{rs}^k}{dt} \\ + \langle p i | \eta^k | j p \rangle \frac{dP_i^k}{dt} \frac{dP_j^k}{dt} + 2 \langle \varepsilon i j | \eta^k | r p \rangle \frac{d\varepsilon_{ij}^k}{dt} \frac{dP_r^k}{dt} = \left\langle \frac{d}{dt} \varepsilon^k(t) | \eta^k | \frac{d}{dt} \varepsilon^k(t) \right\rangle \\ + \left\langle \frac{d}{dt} P^k(t) | \eta^k | \frac{d}{dt} P^k(t) \right\rangle + 2 \left\langle \frac{d}{dt} \varepsilon^k(t) | \eta^k | \frac{d}{dt} P^k(t) \right\rangle = \langle \bar{X}^k | \eta^k | \bar{X}^k \rangle,$$

where

$$| \bar{X}^k \rangle = \left| \frac{d}{dt} \varepsilon^k(t) \right\rangle + \left| \frac{d}{dt} P^k(t) \right\rangle.$$

According to the second principle of thermodynamics $D \geq 0$. The operator η^k is, therefore, positive.

Generalized heat conduction equation

If we differentiate the constitutive equation for S by t and use the invariance of the operators with respect to the translation in time we obtain:

$$\frac{dS(t)}{dt} = \int_{-\infty}^t \langle S | k^{km} | \frac{d\zeta(t')}{dt'} \rangle e^{-\frac{(t-t')}{\tau^{km}}} dt'.$$

We multiply the equation on the left side by T :

$$(38) \quad T \frac{dS}{dt} = \int_{-\infty}^t \langle \zeta(t) | S \rangle \langle S | K^{km} | \frac{d\zeta(t')}{dt'} \rangle e^{-\frac{(t-t')}{\tau^{km}}} dt'.$$

On the other hand

$$(39) \quad T \frac{dS}{dt} = \frac{\delta Q}{dt} + D,$$

where $\frac{\delta Q}{dt}$ is the quantity of heat which the system receives per unit of time.

According to the theory of Onsager

$$(40) \quad \frac{\delta Q}{dt} = \lambda_{ij} \frac{\partial^2 T}{\partial x_i \partial x_j},$$

where λ_{ij} is the tensor of heat condition, and $\frac{\partial T}{\partial x_i}$ are the temperature gradients. After substituting (40) and (39) in (38) we obtain the generalized equation of heat conduction:

$$(41) \quad \lambda_{ij} \frac{\partial^2 T}{\partial x_i \partial x_j} = \int_{-\infty}^t \langle \zeta(t) | S \rangle \langle K^{km} | \frac{d\zeta}{dt'}(t') \rangle e^{-\frac{(t-t')}{\tau^{km}}} dt' - D.$$

Approximation of the constitutive equations

We shall make certain simplifications of the constitutive equations on the basis of physical considerations.

Let us examine the case $\sigma_{ij}(t')=0$, $\mathcal{E}_i(t')=0$, $t' \in (-\infty, t]$. From (22) and (23) we obtain:

$$\begin{aligned} \varepsilon_{ij}(t) &= \int_{-\infty}^t e^{-\frac{(t-t')}{\tau^{km}}} B_{ij}^{km} \Delta T(t') dt'; \\ P_i(t) &= \int_{-\infty}^t e^{-\frac{(t-t')}{\tau^{km}}} F_i^{km} \Delta T(t') dt'; \\ S(t) &= \int_{-\infty}^t e^{-\frac{(t-t')}{\tau^{km}}} L^{km} \Delta T(t') dt'. \end{aligned}$$

As is known, owing to the change of the vibrational energy, the average distance between the atoms and molecules building up the lattice or quasilattice changes. The physical nature of the deformation in this case shows that there should be no inherited effects between $\Delta T(t)$ and $\varepsilon_{ij}(t)$. We will, therefore, take:

$$(42) \quad \varepsilon_{ij}(t) = \int_{-\infty}^t e^{-\frac{t-t'}{\tau^{km}}} B_{ij}^{km} \Delta T(t') dt' = \alpha_{ij} \Delta T(t),$$

(α_{ij} — heat expansion coefficient) which means that the integral is represented only by the singular part.

Analogous reasoning may be made for $P_i(t)$ and $S(t)$:

$$P_i(t) = \int_{-\infty}^t e^{-\frac{t-t'}{\tau^{km}}} F_i^{km} \Delta T(t') dt' = \beta_i \Delta T(t);$$

$$S(t) = \int_{-\infty}^t e^{-\frac{t-t'}{\tau^{km}}} L^{km} \Delta T(t') dt' = C \Delta T(t);$$

β_i — pyroelectric coefficient;

C — constant providing the relation between the temperature change and entropy.

After accounting for (42) in (22) for the constitutive equations we obtain:

$$(43) \quad \begin{aligned} \varepsilon_{ij}(t) - \alpha_{ij} \Delta T(t) &= \int_{-\infty}^t e^{-\frac{t-t'}{\tau^{km}}} (B_{ijrl}^{km} \sigma_{rl}(t') + B_{ijr}^{km} \mathcal{E}_r(t')) dt'; \\ P_i(t) - \beta_i \Delta T(t) &= \int_{-\infty}^t e^{-\frac{t-t'}{\tau^{km}}} (F_{irs}^{km} \sigma_{rs}(t') + F_{ir}^{km} \mathcal{E}_r(t')) dt'; \\ S(t) - C \Delta T(t) &= \int_{-\infty}^t e^{-\frac{t-t'}{\tau^{km}}} (L_{ij}^{km} \sigma_{ij}(t') + L_i^{km} \mathcal{E}_i(t')) dt'. \end{aligned}$$

Let us consider the case $\sigma_{kl}(t') = \text{const}$, $\mathcal{E}_k(t') = \text{const}$, $t' \in (-\infty, t]$, $\tau^{km} \neq \infty$.

From (43) follow the equations

$$(44) \quad \begin{aligned} \varepsilon_{ij}(t) - \alpha_{ij} \Delta T(t) &= B_{ijkl} \sigma_{kl}(t) + B_{ijr} \mathcal{E}_r(t); \\ P_i(t) - \beta_i \Delta T(t) &= F_{ijk} \sigma_{jk}(t) + F_{ij} \mathcal{E}_j(t); \\ S(t) - C \Delta T(t) &= L_{ij} \sigma_{ij}(t) + L_k \mathcal{E}_k(t), \end{aligned}$$

where

$$B_{ijrs} = \sum_{km} B_{ijrs}^{km}, \quad B_{ijr} = \sum_{km} B_{ijr}^{km};$$

$$F_{ijr} = \sum_{km} F_{ijr}^{km}; \quad F_{ij} = \sum_{km} F_{ij}^{km};$$

$$L_{ij} = \sum_{km} L_{ij}^{km}; \quad L_i = \sum_{km} L_i^{km}.$$

From the comparison of (44) with the equations of electroelasticity at $\Delta T = 0$ we obtain [4]:

$B_{ijrs}^{-1} B_{rse} = l_{ijr}$ — piezoelectric tensor;

$\delta_{ij} + F_{ilk} B_{lkr}^{-1} B_{rsj} + F_{ij}$ — static dielectric permeability.

Let us examine the case

$$\varepsilon_{ij}(t) = 0, \quad \Delta T(t) = 0, \quad t' \in (-\infty, \infty).$$

From (43₁) we obtain

$$0 = \int_{-\infty}^t e^{-\frac{t-t'}{\tau^{km}}} B_{ijrl}^{km} (\sigma_{kl}(t') + B_{rlhs}^{-1km} B_{hsn} \mathcal{E}_n(t')) dt'.$$

If we exclude the trivial solution $\sigma_{kl}(t') = 0, \mathcal{E}_n(t') = 0$ we have

$$\sigma_{rl}(t) = -B_{rlhs}^{-1km} B_{hsn}^{km} \mathcal{E}_n(t),$$

which is valid also in the case $\sigma_{kl}(t') = \text{const}, \varepsilon_k(t') = \text{const}, t' \in (-\infty, t]$.

Consequently

$$B_{rlhs}^{-1km} B_{hsn}^{km} = l_{rln} = B_{rlhs}^{-1} B_{hsn}$$

for each k and m .

We substitute the result in (43₁). We obtain:

$$\varepsilon_{ij} - \alpha_{ij} \Delta T(t) = \int_{-\infty}^t e^{-\frac{t-t'}{\tau^{km}}} B_{ijrs}^{km} (\sigma_{rs}(t') + l_{rsi} \mathcal{E}_i(t')) dt'.$$

Let us assume that the dissipative work is due only to the internal mechanical friction

$$D = \eta_{ijrs}^{ks} \frac{d\varepsilon_{ij}^k}{dt} \frac{d\varepsilon_{rs}^k}{dt}.$$

In this case the electric field represents a fully equilibrium thermodynamic force. For each subsystem (from 25) we have:

$$(45) \quad \mathcal{E}_i(t) = E_{ijr}^{kp} \varepsilon_{ij}^k(t) + E_{ij}^{kp} P_j^k(t) + E_i^{np} S^k(t);$$

$$\Delta T(t) = E_{ij}^{ks} \varepsilon_{ij}^k(t) + E_i^{ks} P_i^k(t) + E^{ks} S^k(t).$$

If we assume that the aggregate system consists of subsystems of the same type, from (45) we obtain:

$$(46) \quad \mathcal{E}_i(t) = E_{ijr}^p \varepsilon_{ij}^p(t) + E_{ij}^p P_j^p(t) + E_i^p S(t);$$

$$\Delta T(t) = E_{ij}^s \varepsilon_{ij}^p(t) + E_i^s P_i^p(t) + E^s S(t),$$

where

$$E_{ijr}^p = \sum_k E_{ijr}^{kp}, \quad E_{ij}^p = \sum_k E_{ij}^{kp}, \dots, \quad E^s = \sum_k E^{ks}.$$

From (46) we may exclude $S(t)$ and solve the equation obtained with respect to P_j . As a result is obtained:

$$P_j = A_{ijk} \varepsilon_{jk} + A_{ij} \zeta_j + A_i \Delta T.$$

This equation should be valid, both under the assumptions made and a static regime. From its comparison with the equations of electroelasticity we obtain:

$$A_{ijk} = L_{ijk}, \quad A_{ij} = \chi_{ij}, \quad A_i = \beta_i,$$

where χ is the dielectric susceptibility of the medium.

From equation (46₂) analogously to (41) may be obtained the heat conduction equation

$$\lambda_{ij} \frac{\partial^2 T}{\partial x_i \partial x_j} = C_{sp} \frac{\partial T}{\partial t} - T \frac{E_{ij}^s}{E^s} \frac{d\varepsilon_{ij}}{dt} - T \frac{E_i^s}{E^s} \frac{dP_i}{dt} - D.$$

In form the heat conduction equation, at $P=0$, $D=0$ coincides with the corresponding equation of thermoelasticity [27].

Concluding notes

In the last few years a strongly expressed desire to establish links with physics has been noticeable in the mechanics of continuous media. This is determined above all by the interests of many of the investigators being directed towards the study of mechanical media with complex rheological properties (biomechanics, magnetic hydrodynamics, electroacoustic interactions, relativistic hydrodynamics).

Electric, magnetic and other values take part in the constitutive equations of these media along with the purely mechanical ones. In this manner the appropriate physical equations (electromagnetic, thermodynamic, etc.) should be included in the full set of equations in solving concrete problems. Interest in the mechanics of dissipative media has increased in connection with the ever wider use of polymer materials. The study of the irreversible processes in deformable bodies is done with the aid of non-equilibrium thermodynamics. Besides, the equation of heat conduction also plays a considerable role together with the mechanical equations. This complication of the theory is imposed on the one hand by the variety in the properties of the materials and on the other by our striving for a closer approach to the nature of the objects of real existence.

References

1. Богуславский, Л., А. Ванныков. Органические полупроводники и биополимеры, Наука, 1968.
2. Chynoweth, A. J. Appl. Phys., 27, 78, 1956.
3. Goldsmith, G., J. White, J. Chem. Phys., 31, 1175, 1959.
4. Мэзон, У., Пьезоэлектрические кристаллы и их применения в ультразвуке, Москва, 1952.
5. Rez, J. Cesk. Casopis Fis. 13, 31, 1963.
6. Palonsky, J., P. Douzon, C. Sadron. Compt. Rend., 250, 3414, 1960.
7. Duchesne, J., J. Depireux, A. Bertinchamps, N. Cornet, J. van der Kaa. Nature, 188, 405, 1960.
8. Shamos, M., S. Lavin. Nature, 213, 267, 1967.

9. Pohl, H., A. Rembaum, A. Henry. J. Am. Chem. Soc., 84, 2699, 1962.
10. Гуль, В., Г. Лушейкин, В. Догадкин. ДАН СССР, 149, 302, 1963.
11. Fukada, E., M. Date, N. Hirai. Nature, 211, 1079, 1966.
12. Furukawa, T., Y. Uematsu, K. Asakova, Y. Wada. J. Appl. Polymer Sci., 12, 2678, 1968.
13. Kawaу, H., Japan. J. Appl. Phys., 8, 975, 1969.
14. Бранков, Г., Механика на полимерите. Изд. на БАН, София, 1967.
15. Физическая акустика, II, част Б, Мир, Москва, 1969.
16. Cohen, J., S. Edelman. J. Appl. Phys., 8, 3072, 1971.
17. Edelman, S., L. Crisham, S. Roth, J. Cohen, J. Acoust. Soc. Am., 5, 1040, 1970.
18. Kyame, J. Acoust. Soc. Am., 26, 990, 1954.
19. Porter, W. Modern Foundations of System Engineering. The Macmillan Company, New York. Translated into the Russian: Современные основания общей теории систем, Москва, 1971.
20. Dirac, P., The Principles of Quantum Mechanics, Oxford, Clarendon Press, 1947.
21. Haase R., Thermodynamik der Irreversiblen Prozesse. Darmstadt, 1963.
22. Ziegler, H., Some Extremum Principles in Irreversible Thermodynamics with Application to Continuum Mechanics, in Progress in Solid Mechanics. V .VI, North-Holland Publishing Company, Amsterdam, 1963.
23. Kalman, R., P. Falb, M. Arbib. Topics in Mathematical System Theory. New York, 1969. Russian translation: Очерки по математической теории систем. Мир, Москва, 1971.
24. Coleman, B., W. Noll. Foundation of Linear Viscoelasticity. Rev. Mod. Phys., 33, 1961, 239—249.
25. Coleman, B., W. Noll. An Approximation Theorem for Functionals with Application in Continuum Mechanics. Arch. Ration. Mech. and Analysis, 6, 1960, 355—370.
26. Coleman, B., W. Noll. Simple Fluids Fading Memory, Proc. Intl. Symp. Second-order Effects, Haifa, 1962, p. 530—532.
27. Новацкий, В. Динамические задачи термоупругости. Мир, 1970.