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VARIATIONAL PRINCIPLES FOR COSSERAT ELASTIC SHELLS

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1. Introduction

A broad literature concerns the variational approach to the theory of shells, but mainly it is devoted to simplified models of shells or to approximate theories (i.e., linear theory, nonlinear shallow shells, first approximation of thin elastic shells, etc.). A brief account is given in Szwabovicz [12], where the variational formulation is applied to shells undergoing finite rotations, but small strains, using classical Koiter's shell theory. Fundamental works in the framework of special theories are also those of Valid [13], [14] and Aynola [1].

In our opinion, within the general framework of elastic Cosserat shells, really general variational principles can be derived, containing all the other formulations as particular cases. Basic ideas for such approach can be found in the fundamental memoir by Naghdi [6], as well as in many papers of the same author, devoted to the theory of shells ([7], [8]).

Our chief goal is to generate the analogs of the four fundamental principles of the classical theory of elasticity, namely the energy, Hu-Washizu, Castigliano and Hellinger-Reissner principles, and reformulate them in terms of invariant (under rigid-body displacements) kinematical measures. The second set of principles is based on suitable functionals, defined through functions satisfying the objectivity requirement.

Even if it is well known, it remains useful to remind that we are dealing with hyperelastic shells, which means the existence of a surface energy density. The problem of the existence of a strain energy function is strictly related to Betti's reciprocal theorem, as shown in [2] and [10], and we are not going to enter in such details here.

Then, we deal with the complete paraphernalia of the variational literature and just provide an abridged list of functionals relative to each principle, avoiding explicit boring calculations. For definitions, details and deep exposition of Cosserat shell theory, we refer the reader to Naghdi ([7], [8]), even if our presentation is self-consistent.

In Section 2 the basic kinematic variables are defined, and the balance equations and the appropriate constitutive relations are recorded in a fairly compact form.

In Section 3 functional Hilbert spaces, suitable for our present purpose, are introduced, and the general principles are listed, while the invariant principles are presented in Section 4. Since the procedure of deriving the Euler equations from the corresponding functionals is standard (cfr. e.g. Oden and Reddy [9], and Washizu [15]), we do not enter in such details.

2. Basic laws

2.1. Kinematic relations

Let $\mathbb{R}^3$ be endowed with its natural Euclidean structure. A *placement* of a *Cosserat shell* is a triple $(M, \mathbf{R}, \mathbf{D})$ with M a closed domain in $\mathbb{R}^2$, $\mathbf{R} : M \to \mathbb{R}^3$ an embedding of M, and $\mathbf{D} : M \to \mathbb{R}^3$, the *director field*, is a vector field on $\mathbf{R}(M)$, trasversal to $S = \mathbf{R}(M)$.

As the director field $\mathbf{D}$ is trasversal to S, to maintain a compact notation it is convenient to introduce inclusion and projection operators which allow to write any vector and tensor field on S in a tangential part (with respect to the tangent plane) and a set of components "in the direction of $\mathbf{D}$" (cfr. Gurtin and Murdoch [4] and Naghdi [8]).

We denote by $\mathbf{I}$ the *inclusion operator* $\mathbf{I}(\mathbf{x}) : T_{\mathbf{R}(\mathbf{x})}S \to \mathbb{R}^3$ of the tangent space at $\mathbf{R}(\mathbf{x})$ to S into $\mathbb{R}^3$. A *superficial tensor field* $\mathbf{T}$ on S is a field of linear maps $\mathbf{T}(\mathbf{x}) : T_{\mathbf{R}(\mathbf{x})}S \to \mathbb{R}^3$ for any $\mathbf{x} \in M$, while a *three-dimensional tensor field* on S is a field of linear operators $T_{\mathbf{R}(\mathbf{x})}\mathbb{R}^3 \to \mathbb{R}^3$. We define the *conjugate* $\overline{\mathbf{D}}$ of $\mathbf{D}$ as the unique vector field such that

$$\mathbf{I}^{\mathrm{T}}\overline{\mathbf{D}} = 0, \qquad \overline{\mathbf{D}} \cdot \mathbf{D} = 1, \tag{2.1}$$

so that $\overline{\mathbf{D}} = \frac{1}{\mathbf{D}\cdot\mathbf{n}}\mathbf{n}$, with $\mathbf{n}$ the unit normal to S. The vector field $\overline{\mathbf{D}}$ allows to define a *projection operator* $\mathbb{R}^3 \to T_{\mathbf{R}(\mathbf{x})}S$ with respect to $\mathbf{D}$ by the relation

$$\mathbf{P} = 1 - \mathbf{D} \otimes \overline{\mathbf{D}}, \tag{2.2}$$

related to the inclusion operator by the identity $\mathbf{PI} = \mathbf{I}$.

Finally, we denote by ∇_S and Div_S the surface gradient and divergence on S, characterized by the identity

$$\nabla_S \Phi \partial_{\mathbf{x}} \mathbf{R} = \partial_{\mathbf{x}}(\Phi \circ \mathbf{R}), \tag{2.3}$$

where Φ is any function defined on S.

Let two placements of a Cosserat shell $(M, \mathbf{R}, \mathbf{D})$ and $(M, \mathbf{r}, \mathbf{d})$ be assigned, such that $S = \mathbf{R}(M)$ and $s = \mathbf{r}(M)$ are diffeomorphic through the *deformation* $\mathbf{f} : S \to s$, i.e. $\mathbf{r}(\mathbf{x}) = \mathbf{f}(\mathbf{R}(\mathbf{x}))$, for $\mathbf{x} \in M$; with abuse of notation we continue to denote by the same symbol $\mathbf{d}$ the pull-back $\mathbf{d} \circ \mathbf{f}$ of $\mathbf{d}$ to S: differentiating $\mathbf{f}$ and $\mathbf{d}$ on S, we obtain the superficial tensor fields

$$\mathsf{F} = \nabla_S \mathbf{f}, \qquad \mathsf{G} = \nabla_S \mathbf{d}, \tag{2.4}$$

the *deformation gradient* and the *director gradient.* To maintain the notation compact, we shall work with the three-dimensional invertible tensor field on S

$$\mathsf{F} = \mathsf{F}\mathsf{P} + \mathbf{d} \otimes \overline{\mathbf{D}}, \tag{2.5}$$

such that $\mathbf{d} = \mathsf{F}\mathbf{D}$ and $\mathsf{F} = \mathsf{F}\mathbb{I}$.

We shall extensively use the *displacement* and *director displacement* fields on S

$$\mathbf{u} = \mathbf{r} - \mathbf{R}, \qquad \mathbf{w} = \mathbf{d} - \mathbf{D}, \tag{2.6}$$

and their superficial gradients

$$\mathsf{H} = \nabla_S \mathbf{u}, \qquad \mathsf{K} = \nabla_S \mathbf{w}, \tag{2.7}$$

so that $\mathsf{H} = \mathsf{F} - \mathbb{I}$ and $\mathsf{K} = \mathsf{G} - \nabla_S \mathbf{D}$. More succinctly, let

$$\mathsf{H} = (\nabla_S \mathbf{u})\mathsf{P} + \mathbf{w} \otimes \overline{\mathbf{D}}, \tag{2.8}$$

so that $\mathsf{H} = \mathsf{F} - \mathbf{1}$ and $\mathbf{w} = \mathsf{H}\mathbf{D}$, $\mathsf{H} = \mathsf{H}\mathbb{I}$.

The fundamental objects measuring the deformation of S into s are given by the equivalent lists

$$(\mathsf{F}, \mathbf{d}, \mathsf{G}), \qquad (\mathsf{F}, \mathsf{G}), \qquad (\mathsf{H}, \mathbf{w}, \mathsf{K}), \qquad (\mathsf{H}, \mathsf{K}). \tag{2.9}$$

Finally, we recall that the requirement of objectivity selects kinematic measures invariant under rigid-body displacements:

$$\mathsf{F}^{\mathsf{T}}\mathsf{F}, \qquad \mathsf{F}^{\mathsf{T}}\mathsf{G}, \tag{2.10}$$

or

$$\mathbf{E} := \mathbf{F}^{\mathsf{T}}\mathbf{F} - \mathbf{1}, \qquad \mathbb{U} := \mathbf{F}^{\mathsf{T}}\mathbf{G} - \nabla_S \mathbf{D}, \tag{2.11}$$

with a superposed T denoting the transpose.

2.2. Balance laws

The dynamics of a Cosserat shell with respect to a reference configuration $(M, \mathbf{R}, \mathbf{D})$ is described by two superficial tensor fields on S: $\mathbb{N}$, the *surface stress* and $\mathbb{M}$, the *director stress*, together with a superficial vector field on S: $\mathbf{m}$, the *director moment*; denoting by $\mathbf{b}$ and $\mathbf{l}$ the *external force* and *moment*, we write the balance laws in the form

$$\begin{aligned} &\mathrm{Div}_S \mathbb{N} + \mathbf{b} = 0, \\ &\mathrm{Div}_S \mathbb{M} - \mathbf{m} + \mathbf{l} = 0, \\ &\mathrm{Skw}(\mathbb{N}\mathbf{F}^{\mathsf{T}} - \mathbf{G}\mathbb{M}^{\mathsf{T}} + \mathbf{m} \otimes \mathbf{d}) = 0. \end{aligned} \tag{2.12}$$

In compact form, introducing the three-dimensional tensor field on S

$$\mathbf{N} = \mathbb{N} + \mathbf{m} \otimes \mathbf{D}, \tag{2.13}$$

so that $\mathbb{N} = \mathbf{N}\mathbf{P}^{\mathsf{T}}$, and $\mathbf{m} = \mathbf{N}\overline{\mathbf{D}}$, we may rewrite (2.12) as

$$\begin{aligned} &\mathrm{Div}_S(\mathbf{N}\mathbf{P}^{\mathsf{T}}) + \mathbf{b} = 0, \\ &\mathrm{Div}_S \mathbb{M} - \mathbf{N}\overline{\mathbf{D}} + \mathbf{l} = 0, \\ &\mathrm{Skw}(\mathbf{N}\mathbf{F}^{\mathsf{T}} - \mathbf{G}\mathbb{M}^{\mathsf{T}}) = 0. \end{aligned} \tag{2.14}$$

Moreover, $(2.14)_3$ allows to define a three-dimensional symmetric stress tensor L and a superficial tensor field $\mathbf{Q}$ on S, given by

$$\mathbf{L} = \mathbf{F}^{-1}\mathbf{N} - \mathbf{F}^{-1}\mathbf{G}\mathbb{M}^{\mathsf{T}}\mathbf{F}^{-\mathsf{T}}, \qquad \mathbf{Q} = \mathbf{F}^{-1}\mathbb{M}, \tag{2.15}$$

so that the balance equations may be rewritten in terms of these reduced stresses as

$$\begin{aligned} &\mathrm{Div}_S\{(\mathbf{F}\mathbf{L} + \mathbf{G}\mathbf{Q}^{\mathsf{T}})\mathbf{P}^{\mathsf{T}}\} + \mathbf{b} = 0, \\ &\mathrm{Div}_S(\mathbf{F}\mathbf{Q}) - (\mathbf{F}\mathbf{L} + \mathbf{G}\mathbf{Q}^{\mathsf{T}})\overline{\mathbf{D}} + \mathbf{l} = 0, \\ &\mathbf{L} = \mathbf{L}^{\mathsf{T}}. \end{aligned} \tag{2.16}$$

Equations (2.14) and (2.16) must be supplemented by suitable boundary conditions. Let $\partial S = \partial S_d \cup \partial S_t$ be a partition of the boundary ∂S: then the *mixed problem* for (2.14) requires its solutions to satisfy the conditions

$$\begin{aligned} &(\mathbf{u}, \mathbf{w}) = (\mathbf{u}_0, \mathbf{w}_0) \quad \text{on} \quad \partial S_d, \\ &(\mathbf{N}\mathbf{P}^{\mathsf{T}}\nu, \mathbb{M}\nu) = (\mathbf{n}_0, \mathbf{m}_0) \quad \text{on} \quad \partial S_t, \end{aligned} \tag{2.17}$$

with $\mathbf{u}_0$, $\mathbf{w}_0$, $\mathbf{n}_0$, and $\mathbf{m}_0$ assigned vector fields, and ν the outward unit normal to ∂S (a tangent unit vector to S).

2.3. Constitutive relations

Introducing the *surface energy density* per unit area as a function

$$\varphi = \tilde{\varphi}(|\mathbf{H}, \mathbf{w}, |\mathbf{K}) = \hat{\varphi}(\mathbf{H}, |\mathbf{K}) \tag{2.18}$$

(notice that the surface energy density depends from the reference configuaration), the constitutive functions of the theory have the form

$$|\mathbf{N} = \partial_{|\mathbf{H}}\tilde{\varphi}, \qquad |\mathbf{M} = \partial_{|\mathbf{K}}\tilde{\varphi}, \qquad \mathbf{m} = \partial_{\mathbf{w}}\tilde{\varphi}, \tag{2.19}$$

where, by the identities

$$\begin{cases} \partial_{|\mathbf{H}}\tilde{\varphi} = (\partial_{\mathbf{H}}\hat{\varphi})\mathbf{P}^{\mathsf{T}}, \\ \partial_{\mathbf{w}}\tilde{\varphi} = (\partial_{\mathbf{H}}\hat{\varphi})\overline{\mathbf{D}}, \end{cases} \qquad \partial_{\mathbf{H}}\hat{\varphi} = (\partial_{|\mathbf{H}}\tilde{\varphi})\mathbb{I}^{\mathsf{T}} + \partial_{\mathbf{w}}\tilde{\varphi} \otimes \mathbf{D}, \tag{2.20}$$

we may write

$$\mathbf{N} = \partial_{\mathbf{H}}\hat{\varphi}. \tag{2.21}$$

The surface energy introduced above is not objective, and we are thus led to the representation

$$\varphi = \overline{\varphi}(\mathbf{E}, |\mathbf{U}), \tag{2.22}$$

so that, by the identities

$$\begin{cases} \partial_{\mathbf{H}}\hat{\varphi} = 2\mathbf{F}\,\mathrm{Sym}\,(\partial_{\mathbf{E}}\overline{\varphi}) + \mathbf{G}\,(\partial_{|\mathbf{U}}\overline{\varphi})^{\mathsf{T}}, \\ \partial_{|\mathbf{K}}\hat{\varphi} = \mathbf{F}\partial_{|\mathbf{U}}\overline{\varphi}, \end{cases}$$

$$\begin{cases} 2\,\mathrm{Sym}\,(\partial_{\mathbf{E}}\overline{\varphi}) = \mathbf{F}^{-1}\partial_{\mathbf{H}}\hat{\varphi} - \mathbf{F}^{-1}\mathbf{G}\,(\partial_{|\mathbf{K}}\hat{\varphi})^{\mathsf{T}}\,\mathbf{F}^{-\mathsf{T}}, \\ \partial_{|\mathbf{U}}\overline{\varphi} = \mathbf{F}^{-1}\partial_{|\mathbf{K}}\hat{\varphi}, \end{cases} \tag{2.23}$$

we have

$$\mathbf{L} = 2\,\mathrm{Sym}\,(\partial_{\mathbf{E}}\overline{\varphi}), \qquad \mathbf{Q} = \partial_{|\mathbf{U}}\overline{\varphi}. \tag{2.24}$$

3. The four fundamental principles

For definiteness, we introduce the inner product spaces

(3.1)
$$\begin{aligned}
\mathcal{H} &= \{(\mathbf{u},\mathbf{w}) \in W^{1,2}(S,\mathbb{R}^3) \times W^{1,2}(S,\mathbb{R}^3),\ (\mathbf{u},\mathbf{w}) = (\mathbf{u}_0,\mathbf{w}_0) \text{ on } \partial S_d\},\\
\mathcal{L} &= \{(\mathbf{H},\mathbb{K}) \in L^2(S,\mathrm{Lin}(\mathbb{R}^3,\mathbb{R}^3)) \times L^2(S,\mathrm{Lin}(\mathbb{R}^2,\mathbb{R}^3))\},\\
\mathcal{K} &= \{(\mathbf{N},\mathbb{M}) \in W^{1,2}(S,\mathrm{Lin}(\mathbb{R}^3,\mathbb{R}^3)) \times W^{1,2}(S,\mathrm{Lin}(\mathbb{R}^2,\mathbb{R}^3)),\\
&\quad (\mathbf{N}\mathbf{P}^{\mathsf{T}}\nu, \mathbb{M}\nu) = (\mathbf{n}_0,\mathbf{m}_0) \text{ on } \partial S_t\},
\end{aligned}$$

and the variation spaces

$$\begin{aligned}
\mathcal{H}_0 &= \{(\delta\mathbf{u},\delta\mathbf{w}) \in W^{1,2}(S,\mathbb{R}^3) \times W^{1,2}(S,\mathbb{R}^3),\\
&\quad (\delta\mathbf{u},\delta\mathbf{w}) = (0,0) \text{ on } \partial S_d\},\\
\mathcal{K}_0 &= \{(\delta\mathbf{N},\delta\mathbb{M}) \in W^{1,2}(S,\mathrm{Lin}(\mathbb{R}^3,\mathbb{R}^3)) \times W^{1,2}(S,\mathrm{Lin}(\mathbb{R}^2,\mathbb{R}^3))\\
&\quad (\delta\mathbf{N}\mathbf{P}^{\mathsf{T}}\nu, \delta\mathbb{M}\nu) = (0,0) \text{ on } \partial S_t\},
\end{aligned} \tag{3.2}$$

where $W^{1,2}(S,E)$ is the Sobolev space of square integrable E-valued functions on S, with square integrable generalized first gradient (E a vector space).

It is interesting to note that, viewing the kinematic relations $(2.7)_2$, (2.8) and the balance equations $(2.14)_{1,2}$ as operatorial systems $\mathcal{H} \to \mathcal{L}$ and $\mathcal{K} \to L^2(S,\mathbb{R}^3) \times L^2(S,\mathbb{R}^3)$ respectively, and noting that $\mathcal{K} \subset \mathcal{L}$ and $\mathcal{H} \subset L^2(S,\mathbb{R}^3) \times L^2(S,\mathbb{R}^3)$, then these operators are the formal adjoint of each other, with respect to the duality pairing in $\mathcal{L}$ and $L^2(S,\mathbb{R}^3) \times L^2(S,\mathbb{R}^3)$ (notice in fact that these are pivot spaces for $\mathcal{K}$ and $\mathcal{H}$ respectively). In fact, by the divergence theorem on S,

$$\begin{aligned}
&\int_S \{\mathrm{Div}_S(\mathbf{N}\mathbf{P}^{\mathsf{T}}) \cdot \mathbf{u} + (\mathrm{Div}_S\mathbb{M} - \mathbf{N}\overline{\mathbf{D}}) \cdot \mathbf{w}\}\, da\\
&\quad - \int_{\partial S_t} (\mathbf{N}\mathbf{P}^{\mathsf{T}}\nu \cdot \mathbf{u} + \mathbb{M}\nu \cdot \mathbf{w}) ds\\
&= -\int_S \{\mathbf{N} \cdot (\nabla_S\mathbf{u} + \mathbf{w} \otimes \overline{\mathbf{D}}) + \mathbb{M} \cdot \nabla_S\mathbf{w}\}\, da\\
&\quad + \int_{\partial S_d} (\mathbf{N}\mathbf{P}^{\mathsf{T}}\nu \cdot \mathbf{u} + \mathbb{M}\nu \cdot \mathbf{w}) ds,
\end{aligned} \tag{3.3}$$

where da and ds are the surface and line element on S and ∂S respectively.

In this section we shall tacitly assume, without further mention, that the constitutive functions $\tilde{\varphi}$ and $\hat{\varphi}$ in (2.18) are *compatible with the symmetry relations* $(2.12)_3$ and $(2.14)_3$ respectively.

3.1. The energy principle

The energy principle is the classical minimum principle of mechanics: we define the *deformation energy functional* $W = W(\mathbf{u},\mathbf{w})$ as the functional

on $\mathcal{H}$ given by

$$W(\mathbf{u},\mathbf{w}) = \int_S \tilde{\varphi}(\nabla_S \mathbf{u}, \mathbf{w}, \nabla_S \mathbf{w})\, da - \int_S (\mathbf{b}\cdot\mathbf{u} + \mathbf{l}\cdot\mathbf{w})\, da - \int_{\partial S_t} (\mathbf{n}_0\cdot\mathbf{u} + \mathbf{m}_0\cdot\mathbf{w})\, ds \tag{3.4}$$

Then, granted the kinematical relations $(2.7)_2$, (2.8) and $(2.17)_1$ and the constitutive relations (2.19), the weak solutions of $(2.14)_{1,2}$, $(2.17)_2$ correspond to the Euler equations of (3.4),

$$\begin{cases} \mathrm{Div}_S\left(\partial_{(\nabla_S \mathbf{u})}\tilde{\varphi}\right) + \mathbf{b} = 0, \\ \mathrm{Div}_S\left(\partial_{(\nabla_S \mathbf{w})}\tilde{\varphi}\right) - \partial_{\mathbf{w}}\tilde{\varphi} + \mathbf{l} = 0, \\ \left(\left(\partial_{(\nabla_S \mathbf{u})}\tilde{\varphi}\right)\nu, \left(\partial_{(\nabla_S \mathbf{w})}\tilde{\varphi}\right)\nu\right) = (\mathbf{n}_0, \mathbf{m}_0) \qquad \text{on } \partial S_t, \end{cases} \tag{3.5}$$

which are obtained by computing the Gateaux differential of (3.4)

$$\frac{d}{d\alpha} W(\mathbf{u} + \alpha\delta\mathbf{u}, \mathbf{w} + \alpha\delta\mathbf{w}), \tag{3.6}$$

with $\alpha \in \mathbb{R}$, $(\mathbf{u},\mathbf{w}) \in \mathcal{H}$, and $(\delta\mathbf{u}, \delta\mathbf{w}) \in \mathcal{H}_0$, using divergence-type identities ot the form:

$$\int_S \mathbf{N}\cdot\nabla_S(\delta\mathbf{u})\, da = \int_{\partial S} \mathbf{N}\nu\cdot\delta\mathbf{u}\, ds - \int_S \mathrm{Div}_S\mathbf{N}\cdot\delta\mathbf{u}\, da. \tag{3.7}$$

3.2. The Hu-Washizu principle

The energy principle is suitable for many applications, such as existence proofs, characterization of mathematically "fair" energies, etc., but it is not appropriate whenever a careful computation of the stress field distribution is needed, or in cases of neutral equilibia, when the actual state is not a minimum of the deformation energy, but rather corresponds to a saddle point, and buckling and post-buckling analysis is required in terms of the applied loads and stresses (cfr. Valid [13], [14], Koiter [5] and Pietraszkiewicz and Szwabowicz [11]).

To this purpose, it is necessary to work in terms of the so-called "mixed", or Hellinger-Reissner, and the "complementary energy", or Castigliano, principles.The classical procedure is based on the introduction of the kinematical relations $(2.7)_2$, (2.8) and $(2.17)_1$ as constraints to be imposed on the actual stationary points of (3.4). The corresponding Lagrange multipliers are then identified to the surface stresses and moments, and the resulting variational principle, yielding as Euler equations the full system of balance, constitutive and kinematical relations, is the Hu-Washizu principle.

The usual stability criterion may then be applied to this variational principle, as well as those generated by it (cfr. Valid [23]).

On the inner-product space $\mathcal{H} \times \mathcal{L} \times \mathcal{K}$, define the functional

$$
\begin{aligned}
H(\mathbf{u}, \mathbf{w}, \mathbf{H}, \mathbb{K}, \mathbf{N}, \mathbb{M}) &= \int_S \hat{\varphi}(\mathbf{H}, \mathbb{K})\, da - \int_S (\mathbf{b} \cdot \mathbf{u} + \mathbf{l} \cdot \mathbf{w})\, da \\
&+ \int_S \{\mathbf{N} \cdot ((\nabla_S \mathbf{u})\mathbf{P} + \mathbf{w} \otimes \overline{\mathbf{D}} - \mathbf{H}) + \mathbb{M} \cdot (\nabla_S \mathbf{w} - \mathbb{K})\}\, da \\
&- \int_{\partial S_t} (\mathbf{n}_0 \cdot \mathbf{u} + \mathbf{m}_0 \cdot \mathbf{w})\, ds \\
&- \int_{\partial S_d} \{\mathbf{N}\mathbf{P}^{\mathsf{T}} \nu \cdot (\mathbf{u} - \mathbf{u}_0) + \mathbb{M}\nu \cdot (\mathbf{w} - \mathbf{w}_0)\}\, ds.
\end{aligned} \tag{3.8}
$$

Computing the Gateaux differential of H for variations $((\delta\mathbf{u}, \delta\mathbf{w}), (\delta\mathbf{H}, \delta\mathbb{K}), (\delta\mathbf{N}, \delta\mathbb{M})) \in \mathcal{H}_0 \times \mathcal{L} \times \mathcal{K}_0$, we see that the Euler equations of (3.8) are the kinematical relations, the constitutive relations (2.19), (2.21) and the balance equations $(2.14)_{1,2}$, together with the boundary conditions $(2.17)_2$.

3.3. The Castigliano principle

As noted above, this variational principle is useful whenever numerical computations of the stress fields are required. Besides, it is extremely useful to obtain bounds on the energy as a function of applied stresses and for buckling analysis (cfr. Koiter [5] and Valid [13]).

Consider the linear subspace $\mathcal{D} \subset \mathcal{K}$ of the stress functions $(\mathbf{N}, \mathbb{M})$ identically satisfying the balance laws $(2.14)_{1,2}$, and the corresponding linear subspace $\mathcal{D}_0 \subset \mathcal{K}_0$ of the pairs $(\delta\mathbf{N}, \delta\mathbb{M})$ such that

$$
\mathrm{Div}_S(\delta\mathbf{N}\mathbf{P}^{\mathsf{T}}) = 0, \qquad \mathrm{Div}_S(\delta\mathbb{M}) - \delta\mathbf{N}\overline{\mathbf{D}} = 0, \tag{3.9}
$$

and introdue the *complementary energy function*

$$
\psi = \hat{\psi}(\mathbf{N}, \mathbb{M}) \tag{3.10}
$$

such that the constitutive relations have the form

$$
\mathbf{H} = \partial_{\mathbf{N}} \hat{\psi}, \qquad \mathbb{K} = \partial_{\mathbb{M}} \hat{\psi}, \tag{3.11}
$$

then the *complementary energy functional* on $\mathcal{D}$ is given by

$$
C(\mathbf{N}, \mathbb{M}) = -\int_S \hat{\psi}(\mathbf{N}, \mathbb{M})\, da + \int_{\partial S_d} (\mathbf{N}\mathbf{P}^{\mathsf{T}} \nu \cdot \mathbf{u}_0 + \mathbb{M}\nu \cdot \mathbf{w}_0)\, ds. \tag{3.12}
$$

Using the identities

$$
\begin{aligned}
&\int_S \nabla_S \mathbf{u} \cdot (\delta \mathbf{N} \mathbf{P}^\top)\, da - \int_{\partial S_d} \delta \mathbf{N} \mathbf{P}^\top \nu \cdot \mathbf{u}\, ds = 0, \\
&\int_S (\nabla_S \mathbf{w} \cdot \delta \mathbb{M} + \delta \mathbf{N} \overline{\mathbf{D}} \cdot \mathbf{w})\, da - \int_{\partial S_d} \delta \mathbb{M} \nu \cdot \mathbf{w}\, ds = 0,
\end{aligned} \tag{3.13}
$$

it is immediate to see that, granted the relations (3.10), the Euler equations of (3.12), with respect to variations in $\mathcal{D}_0$, i.e.,

$$
\begin{cases}
\partial_{\mathbf{N}} \hat{\psi} = (\nabla_S \mathbf{u}) \mathbf{P} + \mathbf{w} \otimes \overline{\mathbf{D}}, \\
\partial_{\mathbb{M}} \hat{\psi} = \nabla_S \mathbf{w},
\end{cases} \qquad (\mathbf{u}, \mathbf{w}) = (\mathbf{u}_0, \mathbf{w}_0) \quad \text{on } \partial S_d, \tag{3.14}
$$

are a weak formulation of the kinematic relations $(2.7)_2$, (2.8) and $(2.17)_1$. Notice that, in case the deformation energy $\varphi = \hat{\varphi}(\mathbf{H}, \mathbb{K})$ is convex, the complementary energy functional may be obtained applying the Legendre transformation to the deformation energy functional.

3.4. The Hellinger-Reissner principle

This variational principle is the true mixed principle of mechanics, suitable for buckling and post-buckling analysis. Notice that, as we shall see, the deformation energy of the shell enters the functional as a function of the stresses and moments, given by the complementary energy density.

To obtain this principle, notice that in deriving the complementary-energy functional we restricted to the subspaces of $\mathcal{K}$ and $\mathcal{K}_0$ of stress functions identically satisfying the balance laws. When this assumption is dropped, and the balance equations are introduced as constraints in the complementary principle, we obtain the Hellinger-Reissner functional on $\mathcal{H} \times \mathcal{K}$:

$$
\begin{aligned}
R(\mathbf{u}, \mathbf{w}, \mathbf{N}, \mathbb{M}) = &- \int_S \hat{\psi}(\mathbf{N}, \mathbb{M})\, da + \int_{\partial S_d} (\mathbf{N} \mathbf{P}^\top \nu \cdot \mathbf{u}_0 + \mathbb{M} \nu \cdot \mathbf{w}_0)\, ds, \\
&- \int_S \{\mathbf{u} \cdot (\mathrm{Div}_S(\mathbf{N} \mathbf{P}^\top) + \mathbf{b}) + \mathbf{w} \cdot (\mathrm{Div}_S \mathbb{M} - \mathbf{N} \overline{\mathbf{D}} + \mathbf{l})\}\, da \\
&+ \int_{\partial S_t} \{\mathbf{u} \cdot (\mathbf{N} \mathbf{P}^\top \nu - \mathbf{n}_0) + \mathbf{w} \cdot (\mathbb{M} \nu - \mathbf{m}_0)\}\, ds
\end{aligned} \tag{3.15}
$$

which may be rewritten in the equivalent form

$$
\begin{aligned}
R(\mathbf{u},\mathbf{w},\mathbf{N},\mathbf{IM}) &= -\int_S \hat{\psi}(\mathbf{N},\mathbf{IM})\,da \\
&+ \int_S \{\mathbf{N}\cdot((\nabla_S\mathbf{u})\mathbf{P}+\mathbf{w}\otimes\overline{\mathbf{D}})+\mathbf{IM}\cdot\nabla_S\mathbf{w}\}\,da \\
&- \int_S (\mathbf{b}\cdot\mathbf{u}+\mathbf{l}\cdot\mathbf{w})\,da - \int_{\partial S_t}(\mathbf{u}\cdot\mathbf{n}_0+\mathbf{w}\cdot\mathbf{m}_0)\,ds \\
&- \int_{\partial S_d}\{\mathbf{N}\mathbf{P}^{\mathsf{T}}\nu\cdot(\mathbf{u}-\mathbf{u}_0)+\mathbf{IM}\nu\cdot(\mathbf{w}-\mathbf{w}_0)\}\,ds.
\end{aligned}
\tag{3.16}
$$

Granted the constitutive relations (3.11), the Euler equations of (3.16) are then the weak form of the balance equations $(2.14)_{1,2}$, the kinematical relations $(2.7)_2$, (2.8) (expressed in terms of $(\mathbf{u},\mathbf{w})$ and $(\mathbf{N},\mathbf{IM})$ as in (3.14)), and the boundary conditions (2.17).

4. The invariant principles

In the preceding section we had to require a priori that the constitutive functions satisfy the symmetry relations $(2.12)_3$ or $(2.14)_3$; another approach is possible, requiring from the start that the constitutive functions depend from their arguments throught the invariant combinations (2.11) and (2.15). In this case, the spaces $\mathcal{L}$ and $\mathcal{K}$ are now replaced by

$$
\begin{aligned}
\mathcal{L}^* &= \{(\mathbf{E},\mathbf{IU})\in L^2(S,\mathrm{Sym}(\mathbb{R}^3,\mathbb{R}^3))\times L^2(S,\mathrm{Lin}(\mathbb{R}^2,\mathbb{R}^3))\}. \\
\mathcal{K}^* &= \{(\mathbf{L},\mathbf{Q})\in W^{1,2}(S,\mathrm{Sym}(\mathbb{R}^3,\mathbb{R}^3))\times W^{1,2}(S,\mathrm{Lin}(\mathbb{R}^2,\mathbb{R}^3)), \\
&\quad ((\mathbf{F}\mathbf{L}+\mathbf{G}\mathbf{Q}^{\mathsf{T}})\mathbf{P}^{\mathsf{T}}\nu,\mathbf{F}\mathbf{Q}\nu)=(\mathbf{n}_0,\mathbf{m}_0)\ \text{on}\ \partial S_t\},
\end{aligned}
\tag{4.1}
$$

and the same for the variation space $\mathcal{K}_0$, replaced by

$$
\begin{aligned}
\mathcal{K}_0^* = \{&(\delta\mathbf{L},\delta\mathbf{Q})\in W^{1,2}(S,\mathrm{Sym}(\mathbb{R}^3,\mathbb{R}^3))\times W^{1,2}(S,\mathrm{Lin}(\mathbb{R}^2,\mathbb{R}^3)) \\
&((\mathbf{F}\delta\mathbf{L}+\mathbf{G}\delta\mathbf{Q}^{\mathsf{T}})\mathbf{P}^{\mathsf{T}}\nu,\mathbf{F}\delta\mathbf{Q}\nu)=(0,0)\quad\text{on}\ \partial S_t\}.
\end{aligned}
\tag{4.2}
$$

4.1. The energy principle

The *energy functional* in an invariant setting is the functional W^* on $\mathcal{H}$ defined by

$$
\begin{aligned}
W^*(\mathbf{u},\mathbf{w}) &= \int_S \overline{\varphi}(\mathbf{E}(\mathbf{u},\mathbf{w}),\mathbf{IU}(\mathbf{u},\mathbf{w}))\,da \\
&- \int_S (\mathbf{b}\cdot\mathbf{u}+\mathbf{l}\cdot\mathbf{w})\,da - \int_{\partial S_t}(\mathbf{n}_0\cdot\mathbf{u}+\mathbf{m}_0\cdot\mathbf{w})\,ds.
\end{aligned}
\tag{4.3}
$$

Using the identities (2.23), it is easy to see that, granted the kinematical relations (2.4), (2.11) and $(2.17)_1$ and the constitutive restrictions (2.24), its Euler equations are the balance equations (2.16), $(2.17)_2$.

4.2. The Hu-Washizu principle

The functional W here is replaced by

$$
\begin{aligned}
H^*(\mathbf{u},\mathbf{w},\mathbf{E},\mathbf{IU},\mathbf{L},\mathbf{Q}) &= \int_S \overline{\varphi}(\mathbf{E}(\mathbf{u},\mathbf{w}),\mathbf{IU}(\mathbf{u},\mathbf{w}))\,da \\
&- \int_S (\mathbf{b}\cdot\mathbf{u}+\mathbf{l}\cdot\mathbf{w})\,da \\
&- \int_{\partial S_t} (\mathbf{n}_0\cdot\mathbf{u}+\mathbf{m}_0\cdot\mathbf{w})\,ds \\
&+ \int_S \{\mathbf{L}\cdot(\mathbf{E}(\mathbf{u},\mathbf{w})-\mathbf{E})+\mathbf{Q}\cdot(\mathbf{IU}(\mathbf{u},\mathbf{w})-\mathbf{IU})\}\,da \\
&- \int_{\partial S_d} \{(\mathbf{FL}+\mathbf{GQ}^{\mathsf{T}})\mathbf{P}^{\mathsf{T}}\nu\cdot(\mathbf{u}-\mathbf{u}_0)+\mathbf{FQ}\nu\cdot(\mathbf{w}-\mathbf{w}_0)\}\,ds,
\end{aligned}
\tag{4.4}
$$

defined on $\mathcal{H}\times\mathcal{L}^*\times\mathcal{K}^*$, where $\mathbf{E}(\mathbf{u},\mathbf{w})$ and $\mathbf{IU}(\mathbf{u},\mathbf{w})$ are functions *satisfying identically* the kinematical relations (2.4), (2.11). The Euler equations of (4.4) are the kinematical relations (2.4), (2.11), the constitutive relations (2.24), and the balance laws (2.16), together with the boundary conditions (2.17).

4.3. The Hellinger-Reissner principle

The complementary energy function is here a function on $\mathcal{K}^*$

$$
\psi = \overline{\psi}(\mathbf{L},\mathbf{Q}) \tag{4.5}
$$

such that

$$
\mathbf{E} = \partial_{\mathbf{L}}\overline{\psi}, \qquad \mathbf{IU} = \partial_{\mathbf{Q}}\overline{\psi}. \tag{4.6}
$$

Thus the Hellinger-Reissner functional becomes

$$
\begin{aligned}
R^*(\mathbf{u},\mathbf{w},\mathbf{L},\mathbf{Q}) &= -\int_S \overline{\psi}(\mathbf{L},\mathbf{Q})\,da - \int_S (\mathbf{b}\cdot\mathbf{u}+\mathbf{l}\cdot\mathbf{w})\,da \\
&- \int_{\partial S_t} (\mathbf{n}_0\cdot\mathbf{u}+\mathbf{m}_0\cdot\mathbf{w})\,ds \\
&+ \int_S \{\mathbf{L}\cdot\mathbf{E}(\mathbf{u},\mathbf{w})+\mathbf{Q}\cdot\mathbf{IU}(\mathbf{u},\mathbf{w})\}\,da \\
&- \int_{\partial S_d} \{(\mathbf{FL}+\mathbf{GQ}^{\mathsf{T}})\mathbf{P}^{\mathsf{T}}\nu\cdot(\mathbf{u}-\mathbf{u}_0)+\mathbf{FQ}\nu\cdot(\mathbf{w}-\mathbf{w}_0)\}\,ds,
\end{aligned}
\tag{4.7}
$$

where again $\mathbf{E}(\mathbf{u}, \mathbf{w})$ and $\mathbf{IU}(\mathbf{u}, \mathbf{w})$ are functions *satisfying identically* the kinematical relations (2.4), (2.11).

Granted the constitutive restrictions (4.6), the Euler equations of (4.7) are the kinematical relations (2.4), (2.11) (expressed in terms of $(\mathbf{u}, \mathbf{w})$ and $(\mathbf{L}, \mathbf{Q})$), the balance laws (2.16) and the boundary conditions (2.17). Notice finally that, in case the energy density $\overline{\varphi}$ is a convex function of its arguments, the functional (4.7) may be obtained by applying the Legendre transformation to (4.4).

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