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CONTINUUM MECHANICS

DETERMINATION OF THE FORM FACTOR OF THE SLENDER BEAMS LOADED BY BENDING

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Notation

b, h, B, H	–	dimensions of the beam cross-section [m]
d, D	–	inner and outer tube diameters [m]
m	–	exponent of the designed stress-strain curve
x, y, z	–	rectangular coordinates [m]
A	–	cross-sectional area [m^2]
E	–	Young's modulus [MPa]
K_{PL}	–	form factor – plasticity coefficient
M_Z	–	bending moment [Nm]
N	–	normal force [N]
R	–	radius of curvature of the beam neutral axis [m]
S	–	statical moment of the cross-sectional part [m^3]
W_0	–	bending section modulus [m^3]
ε	–	linear strain
σ	–	normal stress [MPa]

Introduction

To solve bending strength of the beam thick-walled cross-section, it is necessary to respect an effect of plasticity, which occurs beyond the proportional limit. We issue from general conditions of the static equilibrium

$$N(x) = \int_A \sigma . dA$$

(1)

$$M_Z(x) = \int_A \sigma . y . dA.$$

Solving these equations we can obtain the maximum permissable value of the bending moment

$$M_{ZULT} = K_{PL}.W_0.\sigma_{ULT}, \tag{2}$$

where W_0 is bending sectional modulus of the cross-section and the value of the form factor K_{PL} corresponds to the value of ultimate stress σ_{ULT}. We usually take σ_{ULT} in range between yield point R_{P02} and maximum stress R_M. The effect of plastic deformations is very often considered, mainly for very tough materials, to be corresponding to perfectly plastic material. In this case we can obtain the magnitude of the form factor in bending from

$$K_{PL} = \frac{S_D + |S_C|}{W_0}, \tag{3}$$

where S_T and S_D are the static moments of the tensile and compressive parts of the cross-section [3]. The values of the form factor determined from Eq.(3) are very optimistic and in other references they are solved more accurate for practical material characteristics and shapes of the cross-sections, e.g. [1], [2], [3] and [4].

1. General solution

The suggested solution will be assumed as follows

a) The cross-section of the beam is symmetrical.

b) The effects of shear are negligible.

c) Transverse sections of the beam, which are plane and normal before bending, remain plane and normal during bending.

d) The beam material is homogenous and isotropic.

e) The compressive material characteristics are the same as tensional ones.

f) The stress occuring in the outermost fibre equals ultimate tensile or compressive stress.

g) The ratio between elastic and plastic part of strain is constant

$$\Phi = \frac{\varepsilon_{EL}}{\varepsilon_{PL}} = \text{const.} \tag{4}$$

It is evident from the assumption f), that strain of the outermost fibre reaches its ultimate value σ_{ULT}, therefore the ultimate curvature of the beam neutral layer can be obtained from relation

$$\frac{1}{R_{ULT}} = -y_m.\varepsilon_{ULT}. \tag{5}$$

On the basis of this ultimate curvature we can now determine the distribution of strain in the cross-section

$$\varepsilon = \frac{y}{y_m}\varepsilon_{ULT} \tag{6}$$

and the distribution of stress $\sigma(y) = f[\varepsilon(y)]$ by the stress-strain curve respectively. The equations (1) can be upgraded by means of rel.(6) to obtain the position of the neutral axis and to determine the expression of the ultimate bending moment

$$M_{ULT} = \int_A f[\varepsilon(y)].y.dA. \tag{7}$$

If we know relation $\sigma = f[\varepsilon(y)]$, the equations (2) and (6) can be compared, therefore the general expression of the form factor will be obtained

$$K_{PL} = \frac{\int_A f[\varepsilon(y)].y.dA}{\sigma_{ULT}.W_0}. \tag{8}$$

This relation includes the shape of the cross-section and material characteristics.

The stress-strain relation $\sigma - \varepsilon$ is usually written in form

$$\varepsilon = \varepsilon_{EL} + \varepsilon_{PL} = \sigma/E + g(\sigma), \tag{9}$$

where the first part of the equation corresponds to elastic component of strain and the second part is plastic one. To solve relation (8) it is necessary to determine stress σ from (9). For usual forms of function $g(\sigma)$, f.e. [1] , it is very difficult to obtain inverse expression of Rel. (9) and it is necessary to use an iterative solution.

By means of assumption (4) will be

$$\varepsilon(y) = (1 + \Phi).\varepsilon_{PL}(y) = (1 + \Phi).g[\sigma(y)], \tag{10}$$

therefore

$$\sigma(y) = g^{-1}[\varepsilon(y)/(1 + \Phi)] \tag{11}$$

and from Eq.(6)

$$\varepsilon_{PL}(y) = \frac{1}{(1 + \Phi)} \cdot \frac{y}{y_m} \cdot \varepsilon_{ULT}. \tag{12}$$

Substituting (11) into (8) we find

$$K_{PL} = \frac{\int_A g^{-1}[\varepsilon(y)/(1+\Phi)].y.dA}{\sigma_{ULT}.W_0}. \tag{13}$$

If we know function describing the relation $\varepsilon_{PL} = g(\sigma)$, we can solve Eq.(13) directly.

2. Analytical expression of the stress-strain curve

There is a lot of functions in literature describing relation $\sigma - \varepsilon$. For our applications we will choose function by [2] in form

$$\varepsilon = \sigma/E + \varepsilon_{\max}.e^{-n.(1-\sigma/R_M)} \text{ where } n = \frac{\ln(\varepsilon_{\max}/0.002)}{1 - R_{P02}/R_M} \text{ and } \varepsilon_{\max} = \delta. \tag{14}$$

Function (14) describes mechanical properties of duralumins very well, but its part expressing the plastic strain is for next mathematical operations complicated. To simplify this expression, new stress-strain curve is designed in form

$$\varepsilon = \sigma/E + C.\sigma^m, \tag{15}$$

where coefficient C and exponent m are determined for the best correspondence to an experimental results.

For example we can obtain these coefficients by means of comparison with curve (14) on the basis of these differential criteria:

a) The equivalence of both curves is assumed in yield stress R_{P02} and in ultimate tensile stress R_M. After expressing this conditions we obtain

$$C = \frac{\varepsilon_{\max}}{R_M^m} \text{ and } m = \frac{\ln(0.002/\varepsilon_{\max})}{\ln(R_{P02}/R_M)}. \tag{16}$$

b) It is assumed the equivalence of both curves in point corresponding to ultimate tensile stress and the equivalence of the areas under the both ones. Solution of these conditions is

$$C = \frac{\varepsilon_{\max}}{R_M^m} \text{ and } m = \frac{n}{1 - e^{-n}} - 1. \tag{17}$$

c) It is assumed the equivalence of both curves in point corresponding to ultimate tensile stress and the equivalence of the first moments of the areas under the both ones. In this case we obtain

$$C = \frac{\varepsilon_{\max}}{R_M^m} \text{ and } m = \frac{n^2}{e^{-n} + n - 1} - 2. \tag{18}$$

For next solution will be used an inverse expression of dependence $\sigma - \varepsilon_{PL}$ in form $\sigma = C_1 \varepsilon_{PL}^{m_1}$, where coefficients C_1 and m_1 are taken from relations

$$C_1 = C^{-1/m} \quad \text{and} \quad m_1 = 1/m. \tag{19}$$

3. General solution of the form factor

To compare the results obtained by solution on the basis of suggested stress–strain curve (15), it will be made solution by stress–strain curve (2) too, because this expression is used in many countries as a standard one.

3.1. Solution by material curve [2]

Inverse expression of function describing plastic part of stress–strain curve by means of equation (14) is

$$\sigma = R_M.[1 + 1/n.\ln(\varepsilon_{PL}/\varepsilon_{\max})]. \tag{20}$$

Now we can substitute (12) into (20) and next into (13), therefore we obtain

$$K_{PL} = \frac{S_1.K_1 + S_2 K_2}{W_0}, \tag{21}$$

where the characteristics of the cross-section S_1 and S_2 respect influence of plastic deformations and they are determined by expressions

$$S_1 = \int_A |y|.dA, \quad S_2 = \int_A |y|.\ln(|y|).dA \tag{22a}$$

and the other coefficients K_1 and K_2 expressing influence of material characteristics are determined by

$$K_1 = 1 + 1/n.\ln\left[\frac{1}{(1+\Phi)} \cdot \frac{\varepsilon_{ULT}}{\varepsilon_{\max}} \cdot \frac{1}{y_m}\right], \quad k_2 = \frac{1}{n}. \tag{22b}$$

For practical applications it is possible to take relation and $\varepsilon_{ULT}/\varepsilon_{\max} = 1$ and $\Phi = \varepsilon_{el}/\varepsilon_{pl} \ll 1$, therefore

$$K_1 = 1 + 1/n.\ln(1/y_m). \tag{23}$$

3.2. Solution by designed material curve

The mathematical procedure is the same as in previous part. We shall issue from the inverse form of the plastic part of stress-strain curve (15)

$$\sigma = C_1.\varepsilon_{PL}^{m}, \tag{24}$$

where coefficients C_1 and m_1 are determined by (19). Using them, we can directly express the magnitude of the plasticity coefficient in form

$$K_{PL} = 1/W_0 \cdot \left[\frac{1}{(1+\Phi)} \cdot \frac{1}{y_m}\right]^{m_1} \cdot J_{ZN}, \tag{25}$$

where J_{ZN} is the characteristic of the cross-section with influence of plastic deformations and it is determined by

$$J_{ZN} = 2 \cdot \int_{A/2} |y|^{m_1+1}.dA. \tag{26}$$

4. Solution of the form factor of the rectangular section

In this part the form factor will be determined by means of both stress-strain curves from relations (21) and (25) from point of their comparison. These results will be compared with "accurate" iterative solution, what does not use assumption (4).

4.1. Determination of the form factor by relation (21)

The geometrical characteristics of the rectangular cross-section with influence of plastic deformations will be determined by relations (22) and (23)

$$S_1 = b.h/4, \quad S_2 = S_1.[\ln(h/2) - 1/2],$$

$$K_1 = 1 + 1/n.\ln(2/h), \quad K_2 = 1/n, \quad W_0 = b.h/6, \tag{27}$$

where b and h are the depth and highth of the cross-section. After substituting (27) into (21) we can obtain solution of the form factor

$$K_{PL} = 3/2.[1 - 1/(2.n)], \tag{28}$$

where n is a material characteristic by (14).

4.2. Determination of the form factor by relation (25)

In this case is the geometrical characteristic of the rectangular cross-section determined by relation (26)

$$J_{ZN} = \frac{2.b}{m_1 + 2} \cdot \left(\frac{h}{2}\right)^{m_1+2}, \tag{29}$$

therefore the form factor will be

$$K_{PL} = \frac{3}{m_1 + 2}. \tag{30}$$

The view of results, which has been obtained by means of previous procedures are in Fig.1, where the values of the form factor are expressed with respect of material characteristics presented by ratio R_{P02}/R_M. The solution of equation (30) is presented in Fig.1 by three curves, which respect conditions (16), (17) and (18) and they are compared with solution (28) and with "accurate" iterative solution, both ones by stress-strain curve (14).

From comparison of solution results it issue, that the differences among values of the form factors, what have been obtained by curve (14) and designed curve (15) are non significant. The relative difference of the dependences determined by these procedures in view of the "accurate" iterative solution is not greater than 1.5%, hence we can use the designed stress-strain curve (15) and its conditions (17) for the determination of the other kinds form factors of the cross-sections.

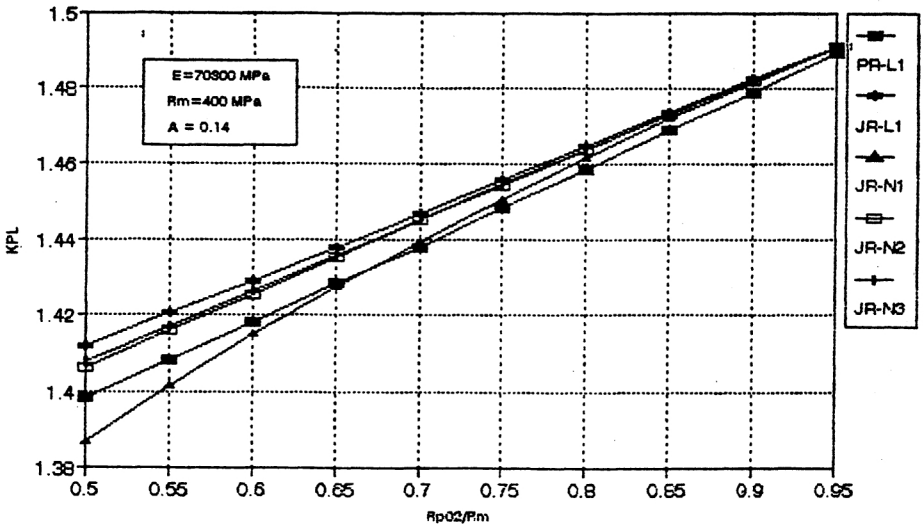


Fig. 1. Form factors. Rectangular section

5. Summary of the form factors of the different kinds of the cross-sections

5.1. "I" section

The cross-section is decomposed into two parts, flanges and web with the same distribution of the strain. Resultant load-carrying capacity equals the sum of the load-carrying capacity both ones, the flanges and the web. The strain distribution along the depth will be after neglecting the elastic strain

$$\varepsilon(y) = \varepsilon_{PL}(y) = \frac{y}{y_m} \cdot \varepsilon_{\max} = \frac{2.y}{H} \cdot \varepsilon_{\max}. \tag{31}$$

By means of equation (31) it is possible to determine values of both ultimate bending moments. For flanges it will be

$$M_{ZULT,FL} = 2 \cdot \int_{h/2}^{H/2} \sigma.y.da = \frac{R_M}{m_1 + 2} \cdot \frac{b.H^2}{2} \cdot \left[1 - \left(\frac{h}{H}\right)^{m_1+2}\right] \tag{32}$$

and for the web we get

$$M_{ZULT,W} = \frac{R_M}{m_1 + 2} \cdot \frac{t_s.h^2}{2} \cdot (h/H)^{m_1+2}. \tag{33}$$

After addition of the both moments from equations (32) and (33) and after division them by bending section modulus we obtain the final value of the form

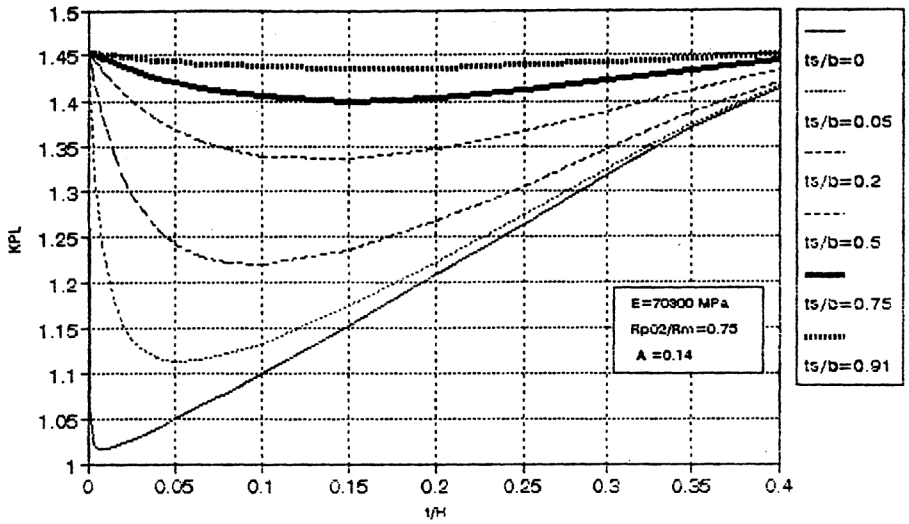


Fig. 2. Form factors. Flanged section

factor

$$K_{PL} = \frac{3}{m_1 + 2} \cdot \frac{1 - \beta.\alpha^{m_1+2}}{1 - \beta\alpha^3}, \tag{34}$$

where

$$\alpha = h/H \quad \text{and} \quad \beta = 1 - t_s/B = b/B.$$

The results of solution of equation (34) are drawn for different geometrical variations in Fig.2.

5.2. A symmetrical trapezoidal cross-section

The same solution procedure as the previous chapter will be used. The cross-section is devided into two bodies — inside rectangular and outside triangular. From condition of the equilibrium and one of the same strain distribution we obtain the final relation for the form factor.

$$K_{PL} = \frac{12}{(m_1 + 2)(m_1 + 3)} \cdot \frac{1 - \alpha^{m_1+3}}{1 - \alpha^4}. \tag{35}$$

The results of solution of equation (35) for different geometrical variants and different material characteristics are in Fig.3.

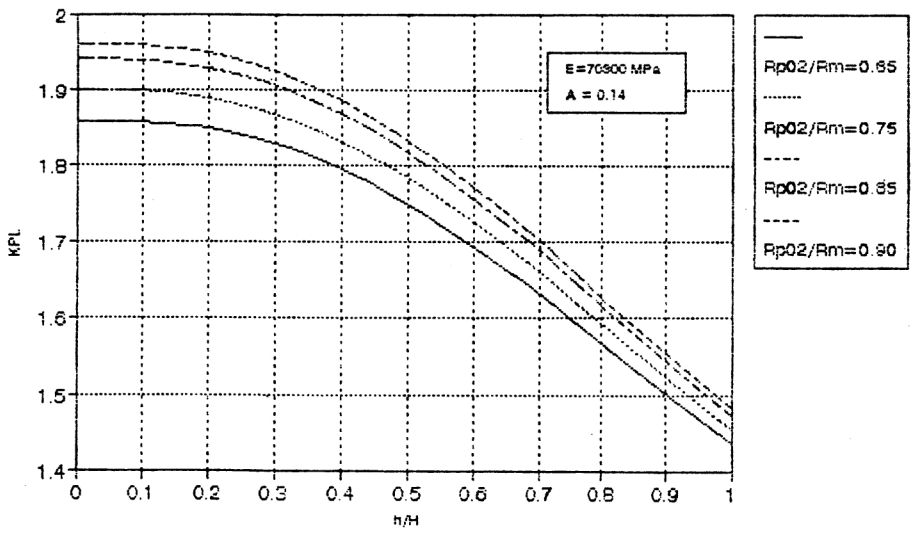


Fig. 3. Form factors. Symmertical trapezoidal section

5.3. A circular section

The value of the ultimate bending moment is determined by the same equations as (31) and (32)

$$M_{ZULT} = 4.R_M.R^3.I, \tag{36}$$

where

$$\bar{y} = y/R \quad \text{and} \quad I = \int_0^1 \bar{y}^{m_1+1}.\sqrt{1-\bar{y}^2}.d\bar{y}. \tag{37}$$

The relation (37) is possible to solve by the series

$$(1 \pm \bar{y})^{1/2} = 1 \pm 1/2.\bar{y} - 1/8.\bar{y}^2 \pm 3/48.\bar{y}^3 - etc. \tag{38}$$

When for next calculation it is used the seventh-degree term (38), from where it is

$$I = \frac{1}{m_1+2} - \frac{0.5}{m_1+4} + \frac{0.125}{m_1+6} - \frac{0.12109}{m_1+8} + \frac{0.041989}{m_1+10}. \tag{39}$$

If we have relation, which determines the value of the ultimate bending moment, we can solve the magnitude of the form factor

$$K_{PL} = \frac{16.I}{\pi}. \tag{40}$$

The results obtained from (40) are drawn in Fig.4 similarly as in Fig.1 in dependence on ratio R_{P02}/R_M. For comparison, the "accurate" iterative solution is drawn too. We can say, that the maximum value of the relative difference is not greater than 1.5%.

From the solution of the circular section we can easy develop the solution of the tube form factor, when the load carrying-capacity of the circular section characterised by diameter D will be decreased a bending moment of inside hole characterised by diameter d. After that is

$$K_{PL} = \frac{16.I}{\pi} \cdot \frac{1-\alpha^{m_1+1}}{1-\alpha^4}. \tag{41}$$

The values of the form factor obtained by (41) is drawn in Fig.5 for different ratios d/D and R_{P02}/R_M. The coefficients C and m of the designed stress-strain curve are determined by conditions (17).

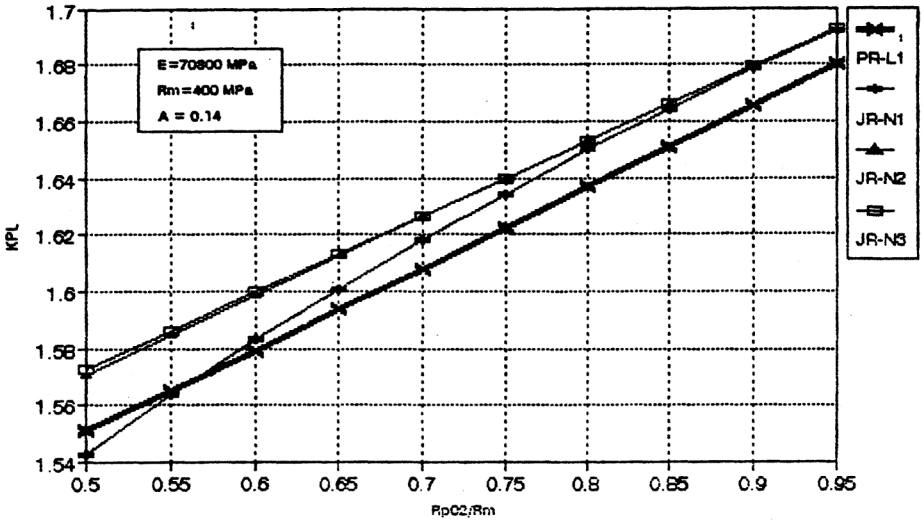


Fig. 4. Form factors. Circular section

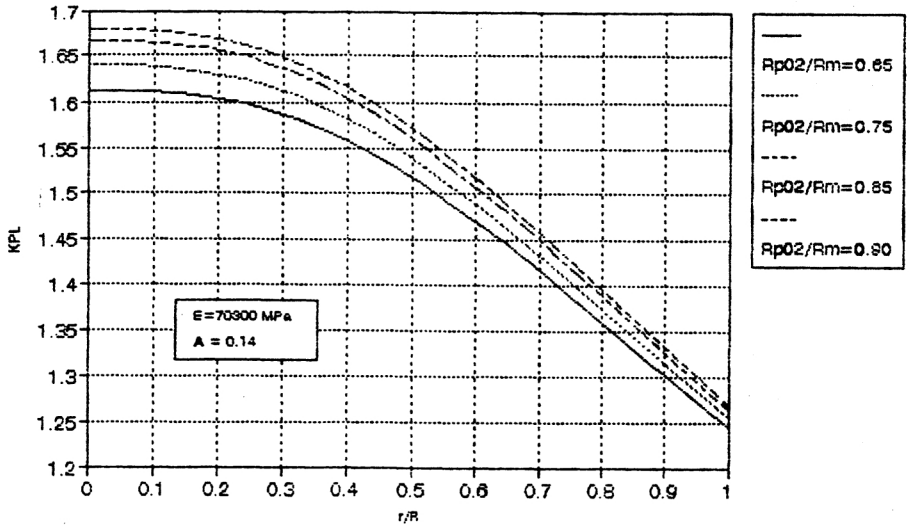


Fig. 5. Form factors. Circular tube

Conclusion

The meaning of this work was to determine the load-carrying capacity of thick-walled cross-sections in bending on base of the simple analytical relations, which respect the mechanical material properties. This analytical

TABLE 1. The values of the form factors for basic shapes of cross-sections.

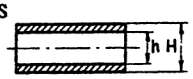
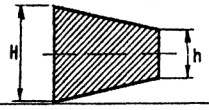
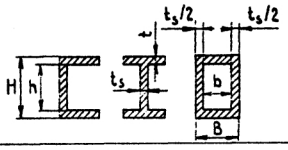
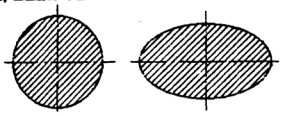
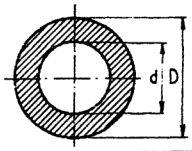
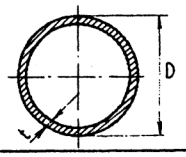
CROSS-SECTION	FORM FACTOR k_{PL}	
	$m_1 = 0,\ \sigma = \sigma_{ULT}$	$m_1 \neq 0,\ \sigma = c_1 \varepsilon_{PL}^{m_1}$
RECTANGULAR SECTION 	1.5	$\frac{3}{m_1+2}$
FLANGES $\alpha = \frac{h}{H}$ 	$1{,}5\frac{1-\alpha^2}{1-\alpha^3}$	$\frac{3}{m_1+2}\frac{1-\alpha^{m_1+2}}{1-\alpha^4}$
TRAPEZOIDAL SECTION 	$2\frac{1-\alpha^3}{1-\alpha^3}$	$\frac{12}{(m_1+2)(m_1+3)}\frac{1-\alpha^{m_1+3}}{1-\alpha^4}$
FLANGED SECTION $\alpha = \frac{h}{H}$ $\beta = \frac{b}{B}$ 	$1{,}5\frac{1-\beta\alpha^2}{1-\beta\alpha^3}$	$\frac{3}{m_1-2}\frac{1-\beta\alpha^{m_1-2}}{1-\beta\alpha^3}$
CIRCLE, ELLIPSE  $I = \frac{1}{m_1+2} - \frac{0.5}{m_1+4} - \frac{0.125}{m_1+6} - \frac{0.12109}{m_1+8} - \frac{0.041989}{m_1+10}$	$\frac{16}{3\pi} \doteq 1.7$	$\frac{16}{\pi} I$
CIRCULAR TUBE $\alpha = \frac{d}{D}$ 	$\frac{16}{3\pi}\frac{1-\alpha^3}{1-\alpha^4}$	$\frac{16}{\pi} I \frac{1-\alpha^{m_1+3}}{1-\alpha^4}$
THIN WALLED TUBE $\frac{t}{d} \to 0$ 	1.279	$\frac{4}{\pi} I (m_1+3)$

Table. 1. The values of the form factors for basic shapes of cross-sections.

solution use the assumption (4), where $\varepsilon_{el}/\varepsilon_{pl} = \text{const}$ and where the elastic strain is neglected. In Tab.1 there is a general view of the form factors of the basic shapes of sections and the solved values are drawn in Fig.2,3 and 5. To compare the results obtained by designed analytical solution with "accurate" iterative solution, there is drawn curve for a rectangular cross-section and a circular one in Fig.1 and Fig.4, which were obtained by both procedures. From these figures we can see, that the load-carrying capacity is a bit higher than the value obtained by "accurate" iterative solution, but relative difference is not greater than 1.5%.

From the previous text is seen, that we can use the designed procedure of calculation of the form factor for the practical determination of the load-carrying capacity of the bended thick-walled sections. This procedure includes an influence of the section shape and the material characteristics, which are not usually included in some other publications.

Legends

$PR - L1$ – "accurate" iterative solution by stress–strain curve (14)
JR – the solution using the assumption (4) by stress–strain curve
$JR - L1$ – by stress–strain curve (14)
$JR - N1$ – by stress–strain curve (15), constants by Eq. (16)
$JR - N2$ – by stress–strain curve (15), constants by Eq. (17)
$JR - N3$ – by stress–strain curve (15), constants by Eq. (18)

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