

ANALYSIS OF SIMULTANEOUS PROPAGATION, SEDIMENTATION AND RESUSPENSION OF CONTAMINANTS BY INTEGRAL TRANSFORM METHODS

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Introduction

Sławomirski et al. [1] investigated the phenomena of propagation of radioactive contaminants and their sedimentation on the earth surface by solving the equation of propagation of radioactive pollution and kinematics of sedimentation: the former corresponds to the balance equation of contaminants which reads:

$$(1) \quad \frac{\partial M(x, t)}{\partial t} + v(x, t) \text{grad} M(x, t) + \frac{\partial R(x, t)}{\partial t} + (\lambda + \chi) M(x, t) = \\ = Q(x, t) + \text{div}[(N + \varepsilon) \text{grad} M(x, t)]$$

where M is concentration of contaminants in atmosphere; R , concentration of contaminants on earth's surface; v , wind velocity vector; Q , intensity of external source of pollutions (e.g. intensity of eruption of radioactive substances from a nuclear plant in which an accident has occurred); λ , radioactive decay coefficient; χ , coefficient of dispersion of pollutions in upper zones of atmosphere; N , coefficient of diffusion of contaminant particles in air; ε , turbulent mixing coefficient for air; x , position vector; and t time.

The kinetics equations of sedimentation may be expressed in various forms, the most common of which are as follows:

a) linear sedimentation kinetics:

$$(2) \quad \frac{\partial R(x, t)}{\partial t} = \alpha(v) M(x, t) - \lambda R(x, t)$$

where α is the sedimentation coefficient dependent generally on magnitude v of velocity vector v ;

b) sedimentation kinetics of the asymptotic type:

$$(3) \quad \frac{\partial R(x, t)}{\partial t} = \alpha(v)[A - R(x, t)]M(x, t) - \lambda R(x, t)$$

where A is the asymptotic value of concentration of contaminants on earth surface;

c) linear kinetics of simultaneous sedimentation and resuspension:

$$(4) \quad \frac{\partial R(x, t)}{\partial t} = \alpha(v)M(x, t) - [\beta(v) + \lambda]R(x, t)$$

where β is resuspension coefficient generally dependent on v . Equation (1) can also be applied to chemical pollution cases with λ being set to zero. Note that the second term of the left side of equation (1) represent the convective transport of contaminants while the second term on the right side expresses the propagation of pollutions by diffusion and turbulent mixing.

When the effects of diffusion and turbulent mixing are small compared with convective contribution, equation (1) may be reduced to:

$$(5) \quad \frac{\partial M(x, t)}{\partial t} + v \frac{\partial M(x, t)}{\partial x} + (\lambda + \chi)M(x, t) + \frac{\partial R(x, t)}{\partial t} = Q(x, t)$$

Its solution can be obtained by means of direct integration methods. In absence of external source, Q , dispersion may be neglected and pollution become non-radioactive (i.e. $\lambda = 0$). The coupled equations (5) and (3) were solved by Trazaska [3,4,5], for a system with cylindrical symmetry. The system of equation (5) and (4) was treated for $\lambda = X = Q = 0$ in reference [2].

Simultaneous convection, sedimentation, and resuspension

Consider a system in which contaminat transfer velocity is constant, the sedimentation parameter, α , is a linear function of velocity, and resuspension parameter β is constant, the equation for sedimentation/resuspension kinetics (4) takes for of:

$$(6) \quad \frac{\partial R(x, t)}{\partial t} = \alpha_o v M(x, t) - (\beta + \lambda)R(x, t)$$

where α_o is a constant value.

The coupled equations (5), (6) are subject to the appropriate initial and boundary conditions:

$$(7) \quad M(x, t)|_{t=0} = M_o(x)$$

$$(8) \quad R(x, t)|_{t=0} = R_o(x)$$

$$(9) \quad M(x, t)|_{x=0} = M_1(t)$$

in which the functions $M_o(x)$, $R_o(x)$, $M_1(x)$ are specified.

Now, equations (5) and (6) are operated on by Carson-Laplace [6,7, 8, 9] transformation with respect to t .

It yields:

$$(10) \quad p\widetilde{M}(x, p) - pM_o(x) + v \frac{\partial \widetilde{M}(x, p)}{\partial x} + (\lambda + \chi)\widetilde{M}(x, p) + p\widetilde{R}(x, p) - pR_o(x) = \widetilde{Q}(x, p)$$

$$(11) \quad p\widetilde{R}(x, p) - pR_o(x) = \alpha_o v \widetilde{M}(x, p) - (\beta + \lambda)\widetilde{R}(x, p)$$

Here, the subscript denotes the Carson-Laplace transformed function and p is the Carson-Laplace variable with respect to t .

A combination of equation (10) and (11) to eliminate $\widetilde{R}$ from the system results in

$$(12) \quad \frac{\partial \widetilde{M}(x, p)}{\partial x} + \left(\frac{p}{v} + \frac{\alpha_o p}{p + \beta + \lambda} + \frac{\chi + \lambda}{v} \right) \widetilde{M}(x, p) = \frac{p}{v} M_o(x) + \frac{p}{v} \frac{\beta + \lambda}{p + \beta + \lambda} R_o(x) + \frac{1}{v} \widetilde{Q}(x, p)$$

Equation (12) is operated on by the Carson-Laplace transform with respect to x . One obtains:

$$(13) \quad s\widetilde{\widetilde{M}}(s, p) - s\widetilde{\widetilde{M}}_1(p) + \left(\frac{p}{v} + \frac{\alpha_o p}{p + \beta + \lambda} + \frac{\chi + \lambda}{v} \right) \widetilde{\widetilde{M}}(s, p) = \frac{p}{v} \widetilde{\widetilde{M}}_o(s) + \frac{\beta + \lambda}{v} \frac{p}{p + \beta + \lambda} \widetilde{\widetilde{R}}_o(s) + \frac{1}{v} \widetilde{\widetilde{Q}}(s, p)$$

wherein, s is the Carson-Laplace variable with respect to x and $\widetilde{\widetilde{M}}_1(p)$ is the Carson-Laplace transform of $M_1(t)$. Equation (13) can be rewritten as:

$$(14) \quad \begin{aligned} \widetilde{\widetilde{M}}(s, p) = & \frac{s}{s + \left(\frac{\alpha_o p}{p + \beta + \lambda} + \frac{p}{v} + \frac{\chi + \lambda}{v} \right)} \widetilde{\widetilde{M}}_1(p) + \\ & + \frac{1}{s} \frac{p}{v} \frac{s}{s + \left(\frac{\alpha_o p}{p + \beta + \lambda} + \frac{p}{v} + \frac{\chi + \lambda}{v} \right)} \widetilde{\widetilde{M}}_o(s) + \\ & + \frac{1}{s} \frac{\beta + \lambda}{v} \frac{p}{p + \beta + \lambda} \frac{s}{s + \left(\frac{\alpha_o p}{p + \beta + \lambda} + \frac{p}{v} + \frac{\chi + \lambda}{v} \right)} \widetilde{\widetilde{R}}_o(s) + \\ & + \frac{1}{s} \frac{1}{v} \frac{s}{s + \left(\frac{\alpha_o p}{p + \beta + \lambda} + \frac{p}{v} + \frac{\chi + \lambda}{v} \right)} \widetilde{\widetilde{Q}}(s, p) \end{aligned}$$

The inverse Carson-Laplace transform of equation (14) with respect to s yields:

$$\begin{aligned}
 \widetilde{M}(x, p) = & \widetilde{M}_1(p) \exp \left[- \left(\frac{\alpha_o p}{p + \beta + \lambda} + \frac{p}{v} + \frac{\chi + \lambda}{v} \right) x \right] + \\
 & + \int_0^x M_o(x - \xi) \frac{p}{v} \cdot \exp \left[- \left(\frac{\alpha_o p}{p + \beta + \lambda} + \frac{p}{v} + \frac{\chi + \lambda}{v} \right) \xi \right] d\xi + \\
 (15) \quad & + \int_0^x R_o(x - \xi) \frac{\beta + \lambda}{v} \frac{p}{p + \beta + \lambda} \cdot \exp \left[- \left(\frac{\alpha_o p}{p + \beta + \lambda} + \frac{p}{v} + \frac{\chi + \lambda}{v} \right) \xi \right] d\xi + \\
 & + \int_0^x \widetilde{Q}(x - \xi, p) \frac{1}{v} \exp \left[- \left(\frac{\alpha_o p}{p + \beta + \lambda} + \frac{p}{v} + \frac{\chi + \lambda}{v} \right) \xi \right] d\xi
 \end{aligned}$$

The inverse Carson -Laplace transform of equation (15) with respect to p produces:

$$\begin{aligned}
 M(x, t) = & c_p^{-1} \left\{ \widetilde{M}_1(p) \exp \left[- \left(\frac{\alpha_o p}{p + \beta + \lambda} + \frac{p}{v} + \frac{\chi + \lambda}{v} \right) x \right] \right\} + \\
 & + \frac{1}{v} \int_0^x M_o(x - \xi) c_p^{-1} \left\{ p \exp \left[- \left(\frac{\alpha_o p}{p + \beta + \lambda} + \frac{p}{v} + \frac{\chi + \lambda}{v} \right) \xi \right] \right\} d\xi + \\
 (16) \quad & + \frac{\beta + \lambda}{v} \int_0^x R_o(x - \xi) c_p^{-1} \left\{ \frac{p}{p + \beta + \lambda} \exp \left[- \left(\frac{\alpha_o p}{p + \beta + \lambda} + \frac{p}{v} + \right. \right. \right. \\
 & \left. \left. \left. + \frac{\chi + \lambda}{v} \right) \xi \right] \right\} d\xi + \\
 & + \frac{1}{v} \int_0^x c_p^{-1} \left\{ \widetilde{Q}(x - \xi, p) \exp \left[- \left(\frac{\alpha_o p}{p + \beta + \lambda} + \frac{p}{v} + \frac{\chi + \lambda}{v} \right) \xi \right] \right\} d\xi
 \end{aligned}$$

After rearranging the terms in equation (16) followed by some mathematical manipulation, it yields:

$$\begin{aligned}
 M(x, t) = & M_1(0) I_o \left[2 \sqrt{(\beta + \lambda) \alpha_o x} \left(t - \frac{x}{v} \right) \right] \exp \left[-(\beta + \lambda) \left(t - \frac{x}{v} \right) \right] \cdot \\
 & \cdot H \left(t - \frac{x}{v} \right) \exp \left[- \left(\alpha_o + \frac{\chi + \lambda}{v} \right) x \right] + \\
 & + \int_0^t \left[\frac{\partial M_1(t - \tau)}{\partial(t - \tau)} + (\beta + \lambda) M_1(t - \tau) \right] I_o \left[2 \sqrt{(\beta + \lambda) \alpha_o x} \left(\tau - \frac{x}{v} \right) \right] \cdot
 \end{aligned}$$

$$\begin{aligned}
& \cdot \exp \left[-(\beta + \lambda) \left(\tau - \frac{x}{v} \right) \right] H \left(\tau - \frac{x}{v} \right) d\tau \exp \left[- \left(\alpha_o + \frac{\chi + \lambda}{v} \right) x \right] + \\
& + \frac{\beta + \lambda}{v} \int_0^x [R_o(x - \xi) + M_o(x - \xi)] I_o \left[2\sqrt{(\beta + \lambda)\alpha_o\xi} \left(t - \frac{\xi}{v} \right) \right] \cdot \\
& \cdot \exp \left[-(\beta + \lambda) \left(t - \frac{\xi}{v} \right) \right] H \left(t - \frac{\xi}{v} \right) \exp \left[- \left(\alpha_o + \frac{\chi + \lambda}{v} \right) \xi \right] d\xi + \\
(17) \quad & + \frac{1}{v} \int_0^x M_o(x - \xi) \frac{\partial}{\partial t} \left\{ \int_0^t \delta(t - \tau) I_o \left[2\sqrt{(\beta + \lambda)\alpha_o\xi} \left(\tau - \frac{\xi}{v} \right) \right] \cdot \right. \\
& \cdot \exp \left[-(\beta + \lambda) \left(\tau - \frac{\xi}{v} \right) \right] H \left(\tau - \frac{\xi}{v} \right) d\tau \left. \right\} \exp \left[- \left(\alpha_o + \frac{\chi + \lambda}{v} \right) \xi \right] d\xi + \\
& + \frac{1}{v} \int_0^x Q(x - \xi, 0) I_o \left[2\sqrt{(\beta + \lambda)\alpha_o\xi} \left(t - \frac{\xi}{v} \right) \right] \cdot \\
& \cdot \exp \left[-(\beta + \lambda) \left(t - \frac{\xi}{v} \right) \right] H \left(t - \frac{\xi}{v} \right) \exp \left[- \left(\alpha_o + \frac{\chi + \lambda}{v} \right) \xi \right] d\xi + \\
& + \frac{1}{v} \int_0^x \int_0^t \left[\frac{\partial Q(x - \xi, t - \tau)}{\partial(t - \tau)} + (\beta + \lambda) Q(x - \xi, t - \tau) \right] \cdot \\
& \cdot I_o \left[2\sqrt{(\beta + \lambda)\alpha_o\xi} \left(\tau - \frac{\xi}{v} \right) \right] \exp \left[-(\beta + \lambda) \left(\tau - \frac{\xi}{v} \right) \right] \cdot \\
& \cdot H \left(\tau - \frac{\xi}{v} \right) d\tau \exp \left[- \left(\alpha_o + \frac{\chi + \lambda}{v} \right) \xi \right] d\xi
\end{aligned}$$

Here $\delta(t)$ is the Dirac delta pseudo-function defined as:

$$(18) \quad \delta(t - \tau) = \begin{cases} \infty & \text{when } t = \tau \\ 0 & \text{when } t \neq \tau \end{cases}$$

$$(19) \quad \int_{-\infty}^{\infty} \delta(t) dt = 1$$

$I_0(y)$ denotes the modified Bessel function. The distribution of concentration of contaminants on the earth surface, $R(x, t)$, may easily be determined solving differential equation (6). The application of direct or integral transform method leads to

$$(20) \quad R(x, t) = R_0(x) \exp [-(\beta + \lambda)t] + \alpha_0 v \int_0^t M(x, \tau) \exp [-(\beta + \lambda)(t - \tau)] d\tau$$

In conclusion, equations (17) and (20) are the solution of the partial differential equation (5) and (6) which govern simultaneous, unsteady, one-dimensional transport, sedimentation and resuspension of radioactive or non-radioactive contaminants.

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