

VIBRATION OF CIRCULAR PLATES ON ELASTIC FOUNDATION WITH IN-PLANE LOADING

R. KURKTCHIEV, I. VAISILOV

1. Introduction

The problem of the stability and the transversal vibrations of thin elastic circular plates has been object of an active interest over 100 years. A great number of problems for circular plates with various geometry, boundary conditions and membrane stress-state have been investigated.

The present work examines the questions of the small free transversal vibration of thin elastic circular plate with constant thickness and with constant membrane radial forces distributed along the plate contour. It is assumed additionally that the plate is rested on Winkler elastic foundation. The influence of the Winkler constant on the plate eigenfrequencies at a given tension or pressure radial loading is to be investigated.

2. Equation of motion

Based on the Sophie Germain-Lagrange equation for thin elastic plates with constant thickness h written in cylindrical coordinates the differential equation of the small variations is as follows:

$$(1) \quad D \nabla^2 \nabla^2 w = q$$

Here

$$(2) \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2},$$

$$(3) \quad D = \frac{Eh^3}{12(1 - \mu^2)}.$$

where E is the Young's modulus and μ is the Poisson's ratio, t denotes time and $w = w(r, \varphi, t)$ is the normal displacement of an arbitrary point on the middle plane. The density q of the normal distributed load in (1) is

$$(4) \quad q = q_m + q_e + q_i,$$

and q_m , q_e , q_i are respectively:

$$(5) \quad q_m = h \left[\sigma_r \frac{\partial^2 w}{\partial r^2} + \sigma_\varphi \left(\frac{1}{r^2} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \varphi^2} \right) + 2\tau \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial w}{\partial r} \right) \right],$$

$$(6) \quad q_e = -kw,$$

$$(7) \quad q_i = \rho h \frac{\partial^2 w}{\partial t^2}.$$

The load(5) represents the membrane equilibrium state of the plate. σ_r , σ_φ and τ in (5) are respectively: normal radial, normal circumfental and shear stress. We confine to the case

$$(8) \quad \sigma_r = \sigma_\varphi = \bar{p} = \text{const.}, \quad \tau = 0$$

Hense

$$(9) \quad q_m = h\bar{p}\nabla^2 w.$$

For $\bar{p} > 0$ ($\bar{p} < 0$) the plate is under tension (pressure).

The component (6) is the reaction of the Winkler foundation, k is the Winkler constant (bed coefficient).

The component (7) is the inertial load, and the mass density is $\rho = \text{const.}$

In the case of charmonic oscillations we assume

$$(10) \quad w(r, \varphi, t) = W(r) \cos(n\varphi) [A \cos(\Omega t) + B \sin(\Omega t)].$$

The constants A and B in (10) can be determined by the initial conditions, Ω is the frequency of the plate free vibrations. The function $W(r)$ defines the vibration's mode. The integer n characterizes the vibration modes in circumferential direction. At $n = 0$ we obtain axisymmetrical vibrations.

After substituting (10) into (1) we obtain the differential equation for $W(r)$:

$$(11) \quad \nabla^2 \nabla^2 w - 2\bar{b} \nabla^2 w - \bar{c} W = 0$$

where

$$(12) \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{n^2}{r^2},$$

$$(13) \quad \bar{b} = \frac{h\bar{p}}{D},$$

$$(14) \quad \bar{c} = \frac{1}{D}(\rho h \Omega^2 - k) > 0.$$

The condition (14) means “soft” elastic foundation.

At tension we have:

$$(15) \quad \bar{p} = p > 0, \quad \bar{b} = b = \frac{ph}{2D} > 0,$$

and the pressure:

$$(16) \quad \bar{p} = -p \ (p > 0), \quad \bar{b} = -b \ (\frac{ph}{2D} > 0).$$

So, we assume that

$$(17) \quad \alpha^2 = \sqrt{b^2 + c} + b = \sqrt{\left(\frac{ph}{2D}\right)^2 + \frac{1}{D}(\rho h \Omega^2 - k)} + \frac{ph}{2D} > 0,$$

$$(18) \quad \beta^2 = \sqrt{b^2 + c} - b = \sqrt{\left(\frac{ph}{2D}\right)^2 + \frac{1}{D}(\rho h \Omega^2 - k)} - \frac{ph}{2D} > 0.$$

The equation (11) for a circular plate without hole and its respective solution are (following [1]):

(i) Plate under tension:

$$(19) \quad (\nabla^2 - \alpha^2)(\nabla^2 + \beta^2)W(r) = 0,$$

$$(20) \quad W(r) = C_1 I_n(\alpha r) + C_2 J_n(\beta r).$$

(ii) Plate under pressure:

$$(21) \quad (\nabla^2 + \alpha^2)(\nabla^2 - \beta^2)W(r) = 0,$$

$$(22) \quad W(r) = C_1 J_n(\alpha r) + C_2 I_n(\beta r),$$

where $J_n(\)$ is a Bessel function of the first kind of order n , and $I_n(\)$ is a modified Bessel function of the first kind of order n . C_1 and C_2 are integral constants.

3. Boundary conditions. Characteristic equation

We assume the plate as elastically clamped along the circle $r = R$. γ is the stiffness of the rotational spring which moulds the support. The boundary conditions are respectively:

$$(23) \quad w|_{r=R} = 0,$$

$$(24) \quad \left[M_r - \gamma \frac{\partial w}{\partial r} \right] \Big|_{r=R} = 0.$$

Taking into account (10) and the expression for the bending moment

$$(25) \quad M_r = D \left[\frac{\partial^2 w}{\partial r^2} + \mu \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \varphi^2} \right) \right]$$

from (23) and (24) we obtain the boundary conditions for the function $W(r)$:

$$(26) \quad W(r) \Big|_{r=R} = 0$$

$$(27) \quad \left[\frac{d^2 W}{dr^2} + \mu \left(\frac{1}{r} \frac{dW}{dr} - \frac{n^2}{r^2} W \right) - \frac{\gamma}{D} \frac{dW}{dr} \right] \Big|_{r=R} = 0.$$

Substituting the solutions (20) and (22) into (26) and (27) we obtain the homogenous linear system of equations for the constants C_1 and C_2 :

$$(28) \quad \begin{cases} d_{11}C_1 + d_{12}C_2 = 0 \\ d_{21}C_1 + d_{22}C_2 = 0. \end{cases}$$

The resulting from this system characteristic equation is

$$(29) \quad d_{11}d_{22} - d_{12}d_{21} = 0.$$

As an example, for the pressured plate we have:

$$(30) \quad d_{11} = J_n(z_\alpha), \quad d_{12} = I_n(z_\beta),$$

$$(31) \quad \begin{aligned} d_{21} &= z_\alpha^2 J_n''(z_\alpha) + \left(\mu - \frac{\gamma R}{D} \right) z_\alpha J_n'(z_\alpha) - \mu n^2 J_n(z_\alpha), \\ d_{22} &= z_\beta^2 I_n''(z_\beta) + \left(\mu - \frac{\gamma R}{D} \right) z_\beta I_n'(z_\beta) - \mu n^2 I_n(z_\beta) \end{aligned}$$

where

$$(32) \quad z_\alpha^2 = (\alpha R)^2 = \sqrt{\tilde{p}^2 + (\tilde{\Omega}^2 - \tilde{k})} + \tilde{p},$$

$$(33) \quad z_\beta^2 = (\beta R)^2 = \sqrt{\tilde{p}^2 + (\tilde{\Omega}^2 - \tilde{k})} - \tilde{p},$$

and

$$(34) \quad \tilde{p} = p \frac{R^2 h}{2D}, \quad \tilde{\Omega} = \Omega R^2 \sqrt{\frac{\rho h}{D}}, \quad \tilde{k} = k \frac{R^4}{D}$$

are dimensionless parameters.

At $\gamma = 0$ the plate becomes simply supported. By $\gamma \rightarrow \infty$ we get an ideal clamping. We consider further only the case of an ideal clamped plate at the boundary $r = R$.

The characteristic equation (29) is:

$$(35) \quad J_n(z_\alpha)z_\beta I_n I'_n(z_\beta) - z_\alpha J'_n(z_\alpha)I_n(z_\beta) = 0.$$

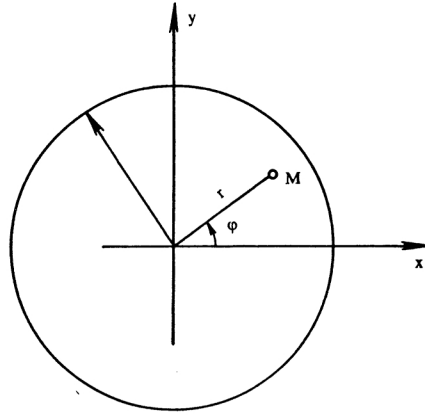


Fig.1 Geometry of the plate

Numerical results. Conclusions

The eigenfrequencies $\tilde{\Omega}$ of an ideally clamped circular plate are evaluated by the numerical solution of the characteristic equation (35), and the results are represented graphically in Fig. 2 and Fig. 3.

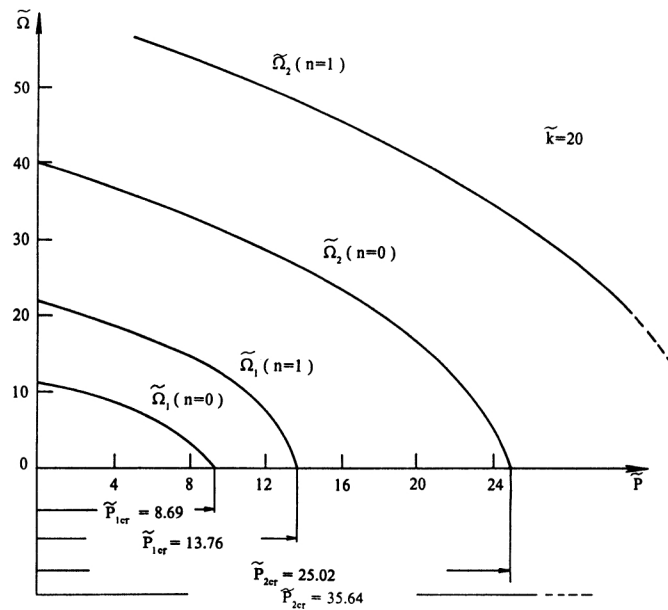


Fig. 2 The frequency as a function of $\tilde{p}$ by $\tilde{k}$ and n constant

The diagrams in Fig. 2 show the dependence of the frequencies $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ with respect to the load parameter $\tilde{p}$ at wave numbers $n = 0$ (axisymmetric mode of vibration), $n = 1$, and a constant Winkler coefficient $\tilde{k} = 20$.

As far as the load $\tilde{p}$ is conservative the curves in Fig. 2 permit to determine the critical value $\tilde{p}_{cr}$, corresponding to the statistical lost of stability of the plate. The critical value $p = \tilde{p}_{cr}$ can be found from the condition:

$$(36) \quad \tilde{\Omega} = \tilde{p}(\tilde{p}_{cr}) = 0.$$

The diagrams in Fig. 3 define the eigenfrequencies $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ as functions of the Winkler coefficient $\tilde{k}$ at wave numbers $n = 0$, $n = 1$ and a constant load $\tilde{p}$. The diagrams show that the frequencies increase together with $\tilde{k}$. Such a behaviour of the plate is easy to be explained. If the coefficient $\tilde{k}$ increases the construction becomes "stiffer" and as far as the mass remains constant, the frequency increases too.

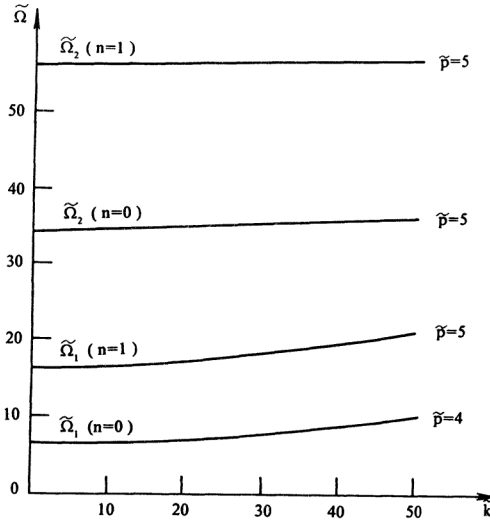


Fig. 3 The frequency $\tilde{\Omega}$ as a function of $\tilde{k}$ by $\tilde{p}$ and n constant

If we put

$$(37) \quad \tilde{\Omega} = \sqrt{\tilde{k}}$$

in (32) and (33) the corresponding values

$$(38) \quad z_\alpha = 2\tilde{p}, \quad z_\beta = 0$$

satisfy the characteristic equation (35) or (29) for each number $n > 0$. Hence, for circular plates based on elastic foundation an additional frequency exists (37) which we call "special". This frequency does not depend on the membrane

load $\tilde{p} < \tilde{p}_{cr}$ and the kind of the support. It is connected only with the elastic foundation.

REFERENCES

- [1] PHYLIPPOV, A. P. Vibration of mechanical systems (in Russian). Naukova dumka, Kiev, 1965.
- [2] WEISENSEL, G. N. Natural frequency information for circular and annular plates. *J. of Sound and Vibr.*, **131**, 1(1989), 129.
- [3] BERT, C. W. Accurate estimation of fundamental frequency for circular plate with in plane loading. *J. of Sound and Vibr.*, **131**, 1(1989), 174.
- [4] WANG, JT. Free vibrations of stepped circular plates of elastic foundation. *J. of Sound and Vibr.*, **159**, 1(1992), 175.
- [5] SHERIF, H. A. Free flexural vibration of clamped circular sandwich plates. *J. of Sound and Vibr.*, **157**, 3(1992), 531.

*Presented at the 7th Congress on Theoretical and Applied
Mechanics, Sofia, August 30 – September 1, 1993
section: Applied Mechanics*